This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

QA
451
A43

PIPPEN'S BOOKSTORE
605 N. Eutaw St.
Baltimore, Marylan
BOOKS BOUGHT & SO

EUCLID'S ELEMENTS

OF

GEOMETRY,

THE SIX FIRST BOOKS.

TO WHICH ARE ADDED,

ELEMENTS OF PLAIN AND SPHERICAL TRIGONOMETRY,

A SYSTEM OF

CONICK SECTIONS,

ELEMENTS OF NATURAL PHILOSOPHY,

AS FAR AS IT RELATES TO ASTRONOMY, ACCORDING TO THE NEWTONIAN SYSTEM,

AND

ELEMENTS OF ASTRONOMY:

WITH NOTES.

BY

THE REV. JOHN ALLEN, A. M.

PROFESSOR OF MATHEMATICKS IN THE UNIVERSITY OF MARYLAND.

"The works of the Lord are great, sought out of all those who have pleasure therein."......Ps. cxi.

BALTIMORE:
PUBLISHED BY CUSHING AND JEWETT.
J. Robinson, printer.
1822.

QA
451
.A43

District of Maryland to wit:

BE IT REMEMBERED, That on this Sixth day of May, in the Forty-sixth year of the Independence of the United States of America, Joseph Cushing and Joseph Jewett of the said District have deposited in this Office the title of a Book, the right whereof they claim as Proprietors, in the words following; to wit:—

"Euclid's Elements of Geometry, the six first Books. To which are added, Elements of Plain and Spherical Trigonometry, a System of Conick Sections, Elements of Natural Philosophy, as far as it relates to Astronomy, according to the Newtonian System, and Elements of Astronomy: with Notes. By the Rev. JOHN ALLEN, A. M. Professor of Mathematicks in the University of Maryland.

"The works of the Lord are great, sought out of all those who have pleasure therein."....Psl. cxi."

In Conformity to the act of Congress of the United States, entitled, "An act for the encouragement of learning, by securing the copies of maps, charts and Books, to the authors and proprietors of such copies during the times therein mentioned." And also to the act, entitled, "An act supplementary to an act, entitled, "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies during the times therein mentioned," and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

PHILIP MOORE,
Clerk of the district of Maryland.

TO THE HONOURABLE

JOHN QUINCY ADAMS,

SECRETARY OF STATE FOR THE UNITED STATES OF AMERICA.

SIR.

In dedicating this work to you, I only pay a tribute of respect, to your many virtues and eminent talents, usefully employed in the service of your country.

In compliance with your particular desire, as well as from my own personal feelings, I think proper to state, that this work was originally intended, to have been dedicated to the late Honourable William Pinkney, Senator of the United States from Maryland, lately the pride and ornament of his country, had not that intention been defeated by his much lamented death.

Wishing, Sir, that you may long continue to devote to the service of your countrymen, those talents and virtues, which they have so justly estimated,

I am, respectfully,

Your most obedient humble servant,

JOHN ALLEN.

Baltimore, May 1st, 1822.

[illegible]

GEORGE BLACKBURN,

Professor of Mathematics, Asbury College, Baltimore.

REVEREND SIR.—The Mathematical work you intend to publish, has my decided approbation. We purpose adopting it in our seminary, and can, with confidence, recommend it to the teachers of that highly important branch of education. You appear to have been particularly careful to avoid the prolixity of most other authors, and, at the same time, have omitted nothing essential to the full demonstration of the propositions:—this will certainly render it more intelligible to the young student, as unnecessary minuteness, and too frequent repetition, only tend to embarrass and confuse him. You have, with much propriety, omitted the 11th and 12th books of Euclid, and inserted in their place, a plain, easy, and concise system of Conick Sections. Your system of Plain and Spherical Trigonometry must render your work very valuable and complete, and cannot fail to insure publick patronage.

JOSEPH WALKER,
Teacher of Mathematicks in the Rev. Dr. Barry's Academy.

SIR.—Having perused the manuscript, which contains the work you intend to publish, on Euclid's Elements, Plain and Spherical Trigonometry, and Conick Sections, I recommend it to the publick, as a useful performance. The Euclidean part is concisely and clearly demonstrated, and freed from the tedious prolixity used by most other writers on Geometry; the properties of the Conick Sections are neatly analysed, brought into a narrow compass, and so blended together, that the pupil can see, at one view, the true analogy, which exists between these curves, and know their properties with ease and dispatch. On the whole, I think it a very suitable work, to be introduced into colleges and seminaries of learning; and for my part, shall give it a decided preference.

OWEN REYNOLDS,
Professor of Mathematicks, Baltimore College.

PREFACE.

The psalmist, in the twenty eighth psalm, gives it as one of the characteristicks of the wicked, that they regarded not the works of the Lord, nor the operation of his hands; and indeed nothing can tend more, to impress the human mind with strong convictions and reverential sentiments of God and his attributes, than an attentive and careful survey of the various works of creation; and among these works, there are none perhaps so well calculated to answer this purpose, as those heavenly bodies, which, by their beauty, order and fitness to answer the purposes for which they appear to have been formed, speak aloud the infinite wisdom, power and goodness of their Almighty Creator; the heavens, as the psalmist expresses it, declaring the glory of God, and the firmament shewing his handy work: thust he contemplation of these things and the laws whereby they are regulated, tend much to strengthen and augment the power of religion over the human mind.

And as the powers of the human intellect appear to be very great on many subjects; so on none is the excellence of these powers, so manifest, as on those of a mathematical and astronomical nature: that eager thirst after knowledge, which is so prevalent among mankind, is very much gratified, on subjects, which afford the most clear and certain conclusions; the planetary system, among its other perfections, performs the office of a most correct and unerring chronometer; can man, who is so curious to pry into the springs and causes of motion of inferior machines of human construction, not wish to discover the causes of the motions of so wonderful a machine, as the planetary system; the workmanship of a being, infinite in wisdom and power?

But besides the tendency of these sciences to improve man's religious state, and employ, advance and gratify his intellectual capacities; they are highly useful to him in his passage through this sublunary state; there is hardly any situation of life in which a man can be placed, wherein he will not find mathematical knowledge useful to him. It is unnecessary to mention particular instances of its utility, which is universally admitted.

And since the utility and interest of mathematical knowledge have, in consequence of the improvements of Newton, been much enhanced; it seems matter of regret, that, since the year 1686, when the first edition of his mathematical principles of Natural Philosophy was published, in the lapse of nearly 140 years, so little should have been done, to diffuse the knowledge of his discoveries, or to render them more generally accessible: among the various causes which have contributed to veil these discoveries from the view of the world, one has been, the general practice of publishing them and the necessary preparatory works, in a language with which comparatively few are acquainted; another has been, the practice of publishing the different preparatory branches, not only very generally in a dead language, but in separate works, thereby rendering the purchase of a greater number of books necessary; and from this mode, it has also happened, that books on those branches, which were of less pressing and general necessity, became extremely scarce and difficult of attainment.

I have therefore, in this work, given all the mathematicks of a synthetical nature, which appeared necessary for understanding these discoveries, and have joined to these parts of Euclid's elements which are most useful and generally read, a system of conick sections, so compressed, that the whole, including many other highly important things, can be afforded at as low a price, as the generality of editions of Euclid, which want much of this momentous matter.

Much advantage has arisen from this method, as in the part, which contained Euclid's elements, it became important, to give, in the form of corollaries or otherwise, whatever might be necessary in subsequent parts of the work; and in those subsequent parts, the citations can generally be made more distinctly, and the reader is not referred to another book, for the authority of any thing advanced; whereas those, who give systems of conick sections, without having Euclid's elements previously delivered in the same book, must be often at a loss, if they cite any thing except Euclid's original propositions, as to the edition of his elements to which they should refer; another advantage from this mode is, that the student of geometry, when the book with which he commences his studies, which is usually Euclid's elements, is put into his hands, should he have taste and talents for such pursuits, which many no doubt will be found to have, will be in possession of the means of advancing himself to very high degrees in this science.

As to the manner in which this work has been executed; the figures, instead of being in plates, are on the page with the matter of the work, this being found by experience to be by far the best mode.

The wording of the propositions throughout Euclid's Elements, plain and spherical trigonometry, and conick sections, are in general terms; in order however to avoid repetition and prolixity, and to render the propositions more clear and explicit, the letters are sometimes added: in which case they are inclosed in parentheses, and the propositions, being read without them, are expressed in general terms.

In Euclid's second book, besides his demonstrations of the nine propositions from the 2nd to the 10th, both inclusive, are given other demonstrations, which were used by Tacquet; of which the three first are said to be from Campanus, the rest from Maurolycus.

In the fifth book of Euclid's Elements, proportional magnitudes are defined by submultiples, instead of Euclid's method by multiples; as this definition varies but little from that given by Mr. Elrington, in his edition of Euclid's elements, I think it proper, to be particular in mentioning how I first came to use it.

Being relieved from my collegiate studies, by obtaining the degree of Bachelor of Arts, in Trinity College, Dublin, Ireland, in February 1784; I began to compose elements of plain geometry, being the substance of the six first books of Euclid's elements, but in an order different from his: as I never did like Euclid's definition of proportional magnitudes, I thought it proper, to read every thing I could meet with on the subject; in reading a long dissertation on proportionals by Tacquet, I found the property, used in this work, as a definition of proportionals, mentioned by him, as belonging to all proportional propositions, which are of no use, unless for demonstrating others; and so arrange the propositions, that all the affections of the same kind can be demonstrated together.

I conclude with expressing my earnest wish and hope, that this work may be highly instrumental, in promoting and diffusing useful and interesting knowledge.

University of Maryland, Baltimore, May 1*st*, 1822.

EUCLID'S ELEMENTS OF GEOMETRY.

GEOMETRY, is that science, whereby we compare together such quantities, as have extension.

OBSERVATIONS.

A *Proposition*, proposes something to be done or demonstrated.

A *Problem*, proposes something to be done, which, when taken for granted as obvious or self-evident, is called a postulate.

A *Theorem*, proposes something to be demonstrated, which, when taken for granted as obvious or self-evident, is called an axiom.

A *Lemma*, is something demonstrated, in order to prove something which follows.

A *Corollary*, is something drawn, as an inference, from a preceding proposition.

A *Scholium*, is a remark or remarks on something which preceded.

The rules of mathematical reasoning, whereby we should be directed in the prosecution of this science, and which have been formed after mature consideration, are few and obvious; and are as follow:

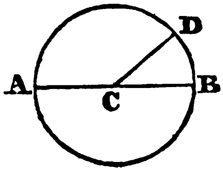
1. The principles assumed, whether practical or theoretical, under the appellation of postulates and axioms, ought to be as few and as simple as possible.
2. Nothing ought to be assumed, in any construction or demonstration, but these principles.

Note.—When in the quotations you meet two numbers, the first shews the proposition, and the second the book. Also Post. denotes Postulate; Ax. Axiom; Def. Definition; B. Book; Constr. Construction; Hyp. Hypothesis, or supposition; contra hyp. contra hypothesin, or contrary to the supposition.

BOOK I.

DEFINITIONS.

1. A *point,* is that which has no part.
2. A *line,* is length without breadth.
3. The *extremes of a line,* are points.
4. A *right line,* is that which lieth equally between its points.
5. A *superficies,* is that which hath length and breadth only.
6. The *extremes of a superficies,* are lines.
7. A *plain superficies,* is that which lieth equally between its lines.
8. The *distance* of two points from each other, is a right line drawn from one of them to the other.
9. A *figure,* is that which is inclosed by one or more boundaries.
10. A *circle,* is a plain figure, bounded by one line (ADBA), every where equally distant from a point (C) within it.



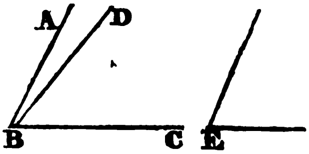
11. That point (C), is called the *centre* of the circle.
12. The bounding line (ADBA), is called its *circumference* or *periphery.*
13. A *diameter* of a circle, is a right line, passing through the centre, and terminated both ways by the circumference, (as AB).
14. A *radius* of a circle is a right line, drawn from the centre to the circumference, (as CD).
15. A *semicircle,* is a figure, contained by a diameter of a circle, and the part of the circumference which is cut off thereby, (as ADB).

16. A *plain angle*, is the inclination of two lines to each other in a plain, which meet together, but are not in the same direction.

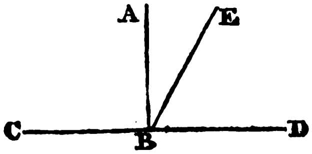
17. A *rectilineal angle*, is the inclination of two right lines, meeting each other, and not being in the same right line.

18. The *legs* of an angle, are the lines, which form the angle.

19. The *vertex* of an angle, is the point, in which the legs meet each other.



An angle is designated either by one letter placed at its vertex, as E; or by three letters, of which the middle one is at the vertex, the other two somewhere in the legs; thus, the angle formed by the lines DB, BC, meeting in B, is called the angle DBC or CBD.



20. When one right line (AB), standing on another (CD), makes the angles (ABC, ABD) on each side of the insisting line (AB) equal, each of these equal angles, are called *right angles*, and the insisting line, is said to be *perpendicular* to the other.

21. The *distance* of a point from a right line, is a perpendicular drawn from the point to the right line.

22. An angle (EBC), which is greater than a right angle, is called *obtuse*.

23. An angle (EBD), which is less than a right angle, is called *acute*.*

---

* Mathematicians have supposed the whole circumference of a circle to be divided in 360 equal parts, called degrees, each degree into 60 equal parts, called minutes, and each minute into 60 equal parts, called seconds, &c. and a circle being described from the vertex of an angle, as a centre, an angle is said to be of as many degrees, minutes, seconds, &c. as are contained in the arch intercepted between the legs of the angle. Thus, see fig. to Def. 10, above, the angle DCB, is said to be of as many degrees, minutes, seconds, &c. as are,

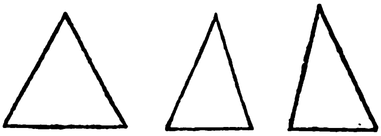
24. A *rectilineal figure,* is a plain one, bounded by right lines.

25. A *triangle,* is a plain figure, bounded by three right lines.

26. A *quadrilateral figure,* or *quadrangle,* is one, bounded by four right lines.

27. Plain figures, bounded by more than four right lines, are called *polygons.*

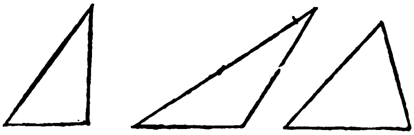
28. Of triangles, those, whose three sides are equal, are called *equilateral.*



29. An *isosceles triangle,* is one, which has only two equal sides.

30. A *scalene triangle,* one, which has three unequal sides.

31. A *right-angled triangle,* is one, which has one right angle.



32. An *obtuse angled triangle,* one, which has one obtuse angle.

33. An *acute-angled triangle,* one, which has three acute angles.

34. *Parallel right lines,* are such as, being in the same plain, would never meet, though ever so much produced both ways.

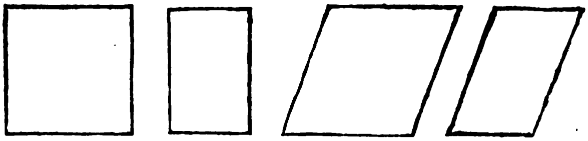
35. Of quadrilateral figures or quadrangles, a *parallelogram,* is one, whose opposite sides are parallel.

36. A *square,* is one, which has all its sides equal, and all its angles right.

---

contained in the arch BD, as will be more fully explained in the tract on plain trigonometry in this work.

This note is inserted, to give beginners a more correct idea of the magnitude of angles, about which they are apt at first to be puzzled. And it is chiefly for the sake of setting them right in this particular, that it was thought expedient, to place the definition of a circle before that of an angle, contrary to the usual practice.



37. An *oblong*, one, which has all its angles right, but not its sides equal.

38. A *rhombus*, one, which is equilateral, but not right angled.

39. A *rhomboid*, one, whose opposite sides and angles are equal, but which is neither equilateral nor right angled.

40. All other quadrilateral figures, besides these, are called *trapeziums*.

POSTULATES.

1. It is required to be granted, that a right line, may be drawn from any point to any other point.

2. That a terminated right line, may be produced at pleasure in a right line.

3. That from any point as a centre, at any distance from that centre, a circle may be described.

AXIOMS.

1. Things, which are equal to the same thing, are equal to each other.

2. If to equals, equals be added, the wholes are equal.

3. If from equals, equals be taken away, the remainders are equal.

4. If to unequals, equals be added, the wholes are unequal, that, which arises from the addition to the greater, being the greater.

5. If from unequals, equals be taken away, the remainders are unequal, that, which arises from the subtraction from the greater, being the greater. And if from equals, unequals be taken away, the remainders are unequal, that, which remains from the subtraction of the greater, being the less.

6. Things, which are double of the same, are equal to each other.

7. Things, which are halves of the same, are equal to each other.

8. Things, which, being applied to each other, do coincide, are equal.

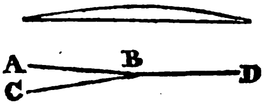
Corollary. The whole is equal to all its parts.

9. The whole is greater than its part.

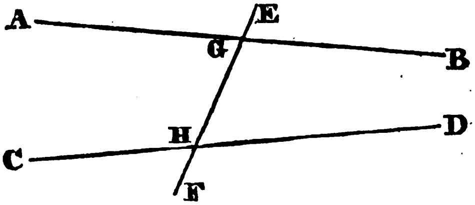
10. Two right lines cannot enclose a space.

11. Two right lines (AB, CB) have not a common segment (BD).

See notes on 10th, 11th, and 12th, ax. and on Prop. 4. of this book.



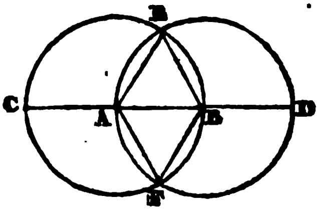
12. If a right line (EF), intersecting two others (AB, CD), make the two interior angles (AGH, GHC) on one side of the intersecting right line, taken together, not equal to the two interior angles (BGH, GHD) on the other side of the intersecting right line, taken together; the right lines (AB, CD), so met by the third, may be so produced towards the part (B, D), on which the interior angles taken together are least, as to meet.



See note to Prop. 29th of this book.

PROPOSITION I. PROBLEM.

Upon a given right finite line (AB), to make an equilateral triangle.



From the centre A, at the distance AB, describe the circle BEC (post. 3). From the centre B, at the distance BA, describe the circle AED (post. 3). Produce AB both ways (post. 2), so as to meet these circles in the points C and D. Then, since the point C is without the circle AED, and the point B within the same circle, and the circles BEC, AED, are continued lines on each side of the right line CD [def. 10], these circles intersect each other on each side of that right line, as in E and F. From one of these intersections E, draw the right lines EA, EB to the extremes of the given right line [Post. 1]. The triangle ABE, which is constituted on the given right line AB, is equilateral.

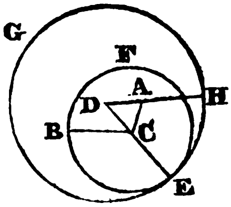
For AE is equal to AB, being both radiuses of the same circle BEC [Def. 10]; and BE is equal to AB, being both radiuses of the same circle AED [Def. 10]; whence AE and BE, being each equal to AB, are equal to each other [Ax. 1]. Therefore AB, AE and BE are equal to each other, and the triangle ABE is equilateral [Def. 28].

Scholium. AF and BF being drawn, the triangle ABF may in like manner be proved to be equilateral.

See note on this proposition.

PROP. II. PROB.

At a given point (A), to put a right line, equal to a given right line (BC).



From the given point A, to either extreme C of the given right line, draw the right line AC [Post. 1]. On AC make the equilateral triangle ADC [1. 1]. From the centre C, at the distance CB, describe the circle EBF [Post. 3], and produce DC to meet its circumference in E [Post. 2]. From the centre D, at the distance DE, describe the circle EGH [Post. 3]. Produce DA to meet its circumference in H [Post. 2]. AH is equal to the given right line BC.

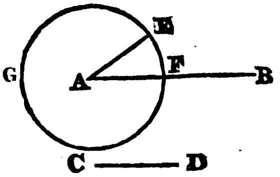
For DH and DE are equal, being radiuses of the same circle EGH [Def. 10]; taking from them the parts DA, DC, which are equal, being sides of the equilateral triangle ADC, the residues AH, CE are equal [Ax. 3]. But BC and CE are equal, being radiuses of the same circle EBF [Def. 10]: therefore AH and BC, being each equal to CE, are equal to each other [Ax. 1]. There is therefore put at the given point A, a right line AH, equal to the given right line BC.

Scholium. The position of the right line AH is varied, according to the extreme of the given right line, to which the right line is drawn from the given point; and also, according to the part of the right line so drawn, to which the triangle is constituted.

PROP. III. PROB.

Two unequal right lines (AB, CD) being given, to cut off from the greater, a part equal to the less.

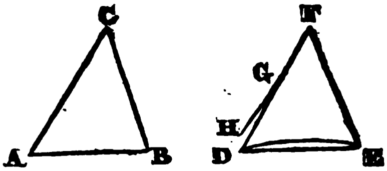
At either extreme A, of the greater of the given right lines, put AE equal to the less CD [2. 1]. From the centre A, at the distance AE, describe the circle EFG [Post. 3], meeting AB in F. The part AF, cut off from AB, is equal to CD.



For AF is equal to AE, being radiuses of the same circle EFG [Def. 10]; and AE is equal to CD [By constr.]; therefore AF and CD are each equal to AE, and therefore to each other [Ax. 1]; and so there is cut off from AB, a part AF, equal to CD.

PROP. IV. THEOREM.

If two triangles (ABC, DEF), have two sides (CA CB), and the angle (ACB) included by them, of one triangle; (ABC), severally equal to two sides (FD, FE), and the angle (DFE) included by them, of the other; the bases or third sides (AB, DE) are equal; as are also, the angles at the bases, opposite to the equal sides (CAB to FDE, and CBA to FED); and the triangles themselves.



For the triangle ABC, being applied to the triangle DEF, so that the point C may coincide in the point F, the right line CA with the right line FD, and the right lines CB, FE be to the same part; the point A would coincide with the point D, for part of CA cannot coincide with FD, and part be without it, as in the direction GH, for then two right lines GH, GD would have a common segment FG, contrary to the 11th axiom; and if the point A went beyond or fell short of the point D, the right lines CA, FD would be unequal [Ax. 9], contrary to the supposition; and because the angles C, F are equal [Hyp.], the right line CB would coincide with FE; and, because, CB, FE are equal [Hyp.], the point B would coincide with the point E.

But the point A coinciding with D, and B with E, the right lines AB, DE would coincide; for, if AB did not coincide, in every part of it, with DE, two right lines would contain a space, contrary to the 10th axiom. Therefore the bases AB, DE are equal [Ax. 8].

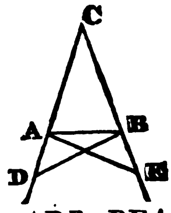
And the legs of the angle CAB, coinciding with those of the angle FDE, and those of the angle CBA, with those of the angle FED; the angles CAB, FDE, as also the angles CBA, FED would coincide, and are therefore equal [Ax. 8].

And the right lines, which contain the triangle ABC, coinciding with the right lines which contain the triangle DEF, the triangles themselves would coincide, and are therefore equal [Ax. 8].

See note on this proposition.

PROP. V. THEOR.

The angles (CAB, CBA), at the base (AB) of an isosceles triangle (ABC), are equal: and, if the equal sides (CA, CB) be produced below the base, the angles under the base (BAD, ABE) are equal.



In either leg, as CA, produced, take any point D; on CB produced, take CE equal to CD [3. 1], and draw DB, AE.

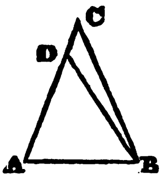
In the triangles DCB, ECA, the sides CD, CE are equal [Constr.], also the sides CB, CA [Hyp.], and the angle C is common to both these triangles; therefore the angle CBD is equal to CAE, the angle CDB to CEA, and BD to AE [4. 1]. Whence, in the triangles ADB, BEA, the angle ADB is equal to BEA, the side DB to AE, and, taking the equals CA, CB, from the equals CD, CE, the side AD to BE [Ax. 3], therefore the angles DAB, ABE, which are the angles under the base, are equal [4. 1].

And, in the same triangles, the angles ABD, BAE are equal; which being taken from the equal angles CBD, CAE, the residues CBA, CAB, which are the angles at the base AB, of the triangle ABC, are equal [Ax. 3].

Corollary.—Hence every equilateral triangle is equiangular. For, whichever side be considered as base, the angles adjacent to it are equal, being opposite equal sides.

PROP. VI. THEOR.

If two angles (CAB, CBA) of a triangle (ABC) be equal, the sides (CA, CB), opposite to them, are equal.



For, if CA and CB be not equal, let one of them, if possible, as CA, be the greater, and take from it AD equal to BC [3. 1], and draw BD.

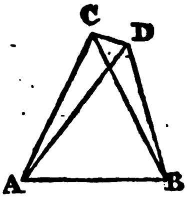
Because, in the triangles DAB, CBA, the sides DA, AB are severally equal to the sides CB, BA, and the angles DAB, CBA included by the equal sides, also equal [Hyp.]; the triangles DAB, CBA are themselves equal [4. 1], a part to the whole, which is absurd [Ax. 9], therefore the sides CA, CB are not unequal, they are therefore equal.

Cor.—Hence every equiangular triangle is equilateral. For, whichever side be considered as base, the angles adjacent to it are equal, and therefore the sides opposite to them.

PROP. VII. THEOR.

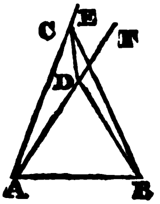
Upon the same base (AB), and on the same side of it, there cannot be two triangles (ACB, ADB), whose conterminous sides are equal, (namely AC to AD, and BC to BD).

For, if possible, let ACB, ADB be such; and first, let the vertex of each fall without the other.



Join CD, and because, in the triangle CAD, the sides AC, AD are equal [Hyp.], the angles ACD, ADC are equal [5. 1.]; but the angle ACD is greater than its part BCD [Ax. 9], therefore the angle ADC is greater than BCD; of course, the angle BDC, which is greater than ADC [Ax. 9], is greater than BCD: but, because the sides BC, BD of the triangle BDC are equal [Hyp.], the angles BDC, BCD are equal [5. 1.]; therefore the angle BDC is both equal to, and greater than, the angle BCD; which is absurd. Therefore two triangles on the same base and same side of it, have not their conterminous sides equal, if the vertex of each fall without the other.

Let now, if possible, the vertex D of either triangle, as ADB, fall within the other.

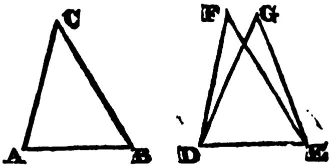


Join CD, and produce AC, AD, as to E and F; and, because in the triangle CAD, the sides AC, AD are equal [Hyp.], the angles ECD, FDC on the other side of the base CD are equal [5. 1]; but the angle BDC is greater than FDC [Ax. 9], and therefore greater also than ECD; and the angle ECD is greater than BCD [Ax. 9]; therefore the angle BDC is greater than BCD: but, in the triangle BCD, because the sides BC, BD are equal [Hyp.], the angles BDC, BCD are equal [5. 1]; therefore the angle BDC is both equal to, and greater than, the angle BCD; which is absurd. Therefore two triangles, on the same base, and same side of it, have not their conterminous sides equal, if the vertex of either fall within the other.

Lastly, let the vertex D, (*see figure to the preceding prop.*) of one, fall on one of the sides AC of the other; and in this case AC, AD are unequal [Ax. 9]. However therefore the vertices of the triangles fall, their conterminous sides are not equal.

PROP. VIII. THEOR.

If two triangles (ABC, DEF), have the two sides (CA, CB) of one, severally equal to the two sides (FD, FE) of the other, and have also their bases (AB, DE) equal; the vertical angles (C, F) are equal.

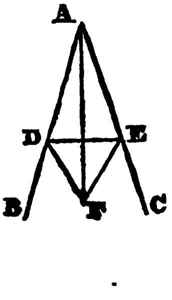


For the triangle ABC being so applied to the triangle DEF, that the point A may coincide with the point D, and the right line AB with DE, the triangles ABC, DFE being to the same part, the point B would coincide with E, because AB is equal to DE; and the whole triangle ACB would coincide with the triangle DFE, for if it should have a different position, as DGE, there would be on DE, and on the same side of it, two triangles, with equal conterminous sides, which is absurd [7. 1]; therefore the sides CA, CB would coincide with the sides FD, FE, and the angle C with the angle F, which angles are therefore equal [Ax. 8].

Corollary.—Because CA, CB and the included angle C, are severally equal to FB, FE and the included angle F, the remaining angles A, B of the triangle ACB, are severally equal to the remaining angles FDE, FED of the triangle DFE, and also the triangles themselves.

PROP. IX. PROB.

To bisect, or divide into two equal parts, a given rectilineal angle (BAC).



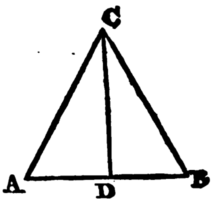
Take any point D in AB, and, on AC, take AE equal to AD [3. 1]; draw DE, on which make the equilateral triangle DFE [1. 1]; draw AF, which bisects the given angle BAC.

For, in the triangles ADF, AEF, the sides AD, AE are equal [Constr.], AF common, and the bases DF, EF also equal [Constr.]; therefore the angles DAF, EAF are equal [8. 1], and of course the given angle BAC is bisected by the right line AF.

Cor.—By the aid of this proposition, an angle may be divided into 2, 4, 8, 16, &c. equal parts, by repeatedly bisecting the several parts.

PROP. X. PROB.

To bisect a given finite right line (AB).

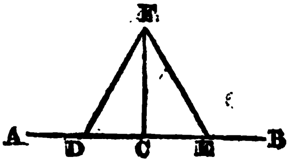


On AB make the equilateral triangle ABC [1. 1], bisect the angle ACB by the right line CD, meeting AB in D [9. 1]. AB is bisected in D.

For, in the triangles ACD, BCD, AC and BC are equal [Constr.], CD common, and the angles ACD, BCD also equal [Constr.], therefore the bases AD, DB are equal [4. 1], and so the right line AB is bisected in D.

PROP. XI. PROB.

To draw a right line perpendicular to a given right line (AB), from a given point (C) therein.

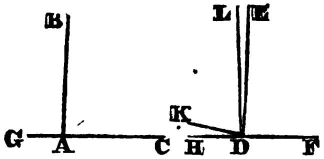


Take any point D in AC, on CB take CE equal to CD [3. 1], and on DE make the equilateral triangle DFE [1. 1]; draw CF, which is perpendicular to AB.

For, in the triangles DCF, ECF, the sides DC, CE, are equal [Constr.], CF common, and the bases DF, EF also equal [Constr.]; therefore the angles DCF, ECF opposite the equal bases are equal [8. 1], and so CF is perpendicular to AB [Def. 20].

Theorem.—All right angles [as BAC, EDF,] are equal to each other.

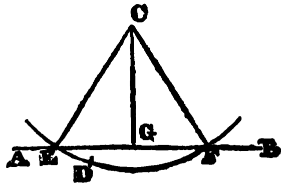
Let CA and FD be produced, as to G and H [Post. 2].



The angle BAC being applied to the angle EDF, so that the point A may coincide with the point D, and the right line AC with DF, the right line AG would coincide with DH, for if AG did not coincide with DH, but had a different situation, as DK, two right lines HD, KD would have a common segment DF, which is absurd [Ax. 11]; also AB would coincide with DE, for if not, let it, if possible, fall on either side of DE, as towards H, in the right line DL; then because EDF is equal to EDH (Def. 20), and LDF greater than EDF (Ax. 9), LDF is greater than EDH; whence, EDH being greater than LDH (Ax. 9), LDF is greater than LDH; but the angle BAC is equal to LDF (Hyp.), and BAG to LDH (Hyp.), therefore BAC is greater than BAG, which is absurd, BAC being, by supposition, a right angle, and therefore the angles BAC, BAG equal to each other (Def. 20). A like absurdity would follow, if AB were to fall on the other side of DE towards F. Therefore AB coincides with DE, and so the angles BAC, EDF coincide, and are of course equal [Ax. 8].

PROP. XII. PROB.

From a given point (C), without a given right line (AB), producible at pleasure, to draw a perpendicular to it.



Let any point whatever, as D, be taken on the other side of AB with respect to the given point C, and from the centre C, at the distance CD, let the circle EDF be described (Post. 3); which, because the points C, D are on different sides of the right line AB, meets the same right line, produced if necessary, in two points, as E and F. Bisect EF in

G (10. 1), and join CG, which is the perpendicular required.

For, CE, CF being drawn, in the triangles CGE, CGF, the sides GE, GF are equal, by construction, CG common, and the bases CE, CF equal, being radiuses of the same circle EDF (Def. 10); therefore the angles CGE, CGF are equal (8. 1), and of course CG is perpendicular to AB (Def. 20).

PROP. XIII. THEOR.

The angles (ABC, ABD), which one right line (AB), makes with another (CD), on one side of it, are together equal to two right angles.

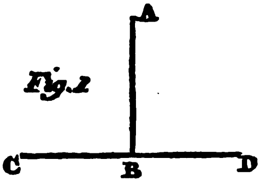


Fig. 1

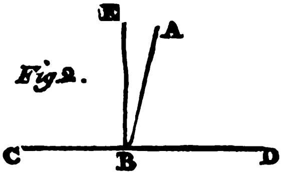


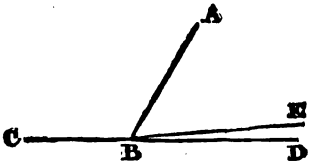
Fig. 2.

If the angles ABC, ABD be equal, (see figure 1), they are both right angles (Def. 20), and therefore together equal to two right angles; if unequal, (see fig. 2), draw BE perpendicular to CD (11. 1), and EBC, EBD are both right angles (Constr.), and therefore together equal to two right angles; but, because the angle ABC is equal to all its parts, the two angles ABE, EBC taken together (Cor. Ax.8), if to each be added the angle ABD, the two angles ABC, ABD are together equal to the three angles CBE, EBA, ABD (Ax. 2); again, because the whole angle EBD is equal to all its parts EBA, ABD (Cor. Ax. 8), if to each CBE be added, the two angles CBE, EBD, are together equal to the three angles CBE, EBA, ABD, (Ax. 2): whence, the two angles CBA, ABD, and the two angles CBE, EBD being each equal to the three angles CBE, EBA, ABD, are equal to each other (Ax. 1): but CBE, EBD are right angles (Constr.), and therefore together equal to two right angles, therefore CBA, ABD are also together equal to two right angles.

Cor.—If more right lines stand on the same right line (CD), on the same side of it, and at the same point (B), not being an extreme; they make angles equal to two right angles.

PROP. XIV. THEOR.

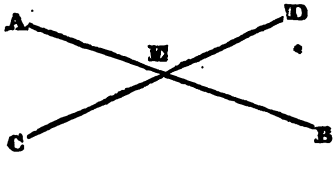
Two right lines (CB, BD), which, at the same point (B), and on different sides of a right line (AB), make with it, the adjacent angles (CBA, DBA), together equal to two right angles, are in the same right line.



If BD be not in the same right line with CB, let some other right line, as BE, be the production of CB; and, because the right line AB falls on the right line CBE, the angles CBA, ABE are equal to two right angles (13. 1); but the angles CBA, ABD are, by supposition, equal to two right angles; therefore the angles CBA, ABE together, and the angles CBA, ABD together, being each equal to two right angles, and all right angles being equal to each other (Theor. at 11. 1), are equal (Ax. 1); taking from each the common angle CBA, the remaining angles ABE, ABD are equal (Ax. 3), part and whole, which is absurd (Ax. 9): therefore BE is not the continuation of CB. In like manner it may be shewn, that no other right line, but BD, can be the continuation of CB; therefore BD is that continuation, and CB, BD are in the same right line.

PROP. XV. THEOR.

If two right lines (AB, CD) intersect each other, the vertical or opposite angles are equal; (AEC to BED, and AED to CEB,).



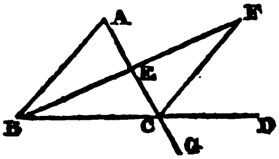
The angles AEC, AED, which AE makes with CD, are equal to two right angles [13. 1.] also the angles AED, DEB, which DE makes with AB, are equal to two right angles [13. 1]; therefore the angles AEC, AED together, are equal to AED, DEB together (Theor. at 11. 1. and Ax. 1.): taking away the common angle AED, the remaining angles AEC and DEB are equal (Ax. 3). In like manner, the angles AED, CEB may be proved equal.

Cor. 1—Two intersecting right lines (AB, CD), make angles, equal to four right angles.

Cor. 2—If more right lines be drawn to the point, wherein two right lines intersect each other, all the angles taken together, are equal to four right angles.

PROP. XVI. THEOR.

If any side (BC) of a triangle (ABC) be produced, the exterior angle (ACD) is greater than either of the interior remote angles (A or ABC)

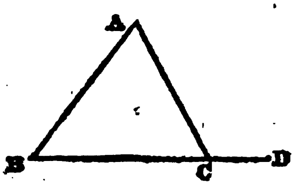


Bisect AC in E (10. 1), join BE, on which produced take EF equal to BE (3. 1), and join CF.

In the triangles AEB, CEF, the sides CE, EF are severally equal to AE, EB [Constr.], and the angles CEF, AEB, being vertical angles, are equal (15. 1); therefore the angles ECF and A are equal (4. 1); whence ACD, being greater than ECF [Ax. 9], is also greater than its equal A. In like manner, if AC be produced, as to G, the angle BCG may be proved to be greater than ABC; and therefore ACD, which is equal to BCG (15. 1), is also greater than ABC.

PROP. XVII. THEOR.

Any two angles of a triangle (ABC), are together less than two right angles.



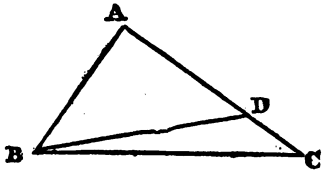
Produce any side BC, as to D, and the exterior angle ACD of the triangle ABC is greater than the interior remote angle B (16. 1); adding to each the angle ACB, the angles ACD, ACB together, are greater than the angles B and

ACB together [Ax. 4]; but the angles ACD, ACB, which AC makes with BD, are together equal to two right angles (13. 1), therefore the angles B and ACB are together less than two right angles. In like manner it may be shown, that any other two angles of the triangle ABC are less than two right angles.

Cor.—If, in any triangle, one angle be obtuse or right, both the others are acute; and if two angles be equal, they are both acute.

PROP. XVIII. THEOR.

If two sides (AB, AC), of a triangle (ABC), be unequal; the angle (ABC), opposite the greater side (AC), is greater than that (C), opposite the less (AB).

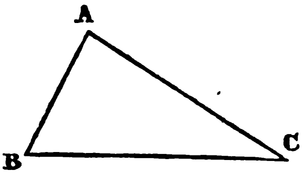


From the greater side AC, take away AD, equal and conterminous to the less AB [3. 1], and join BD.

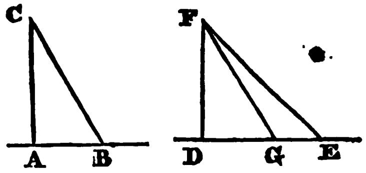
Because the triangle ABD is isosceles, the angles ABD, ADB are equal [5. 1]; but the external angle ADB of the triangle BDC is greater than the internal remote angle C, [16. 1], therefore ABD is greater than C; of course ABC, which is greater than ABD [Ax. 9], is also greater than C.

PROP. XIX. PROB.

If two angles (B, C), of a triangle (ABC), be unequal; the side (AC) opposite the greater angle (B), is greater than that (AB), opposite the less (C).



If AC be not greater than AB, it is either equal to or less than it; it is not equal to AB, for then the angles ABC, ACB would be equal [5. 1], contrary to the supposition; it is not less than AB, for then the angle B would be less than the angle C [18. 1], which is also contrary to the supposition.



Cor. 1.—A perpendicular (CA), drawn from any point (C), to any right line (AB), is less than any other right line (CB), drawn from the same point, to the same right line.

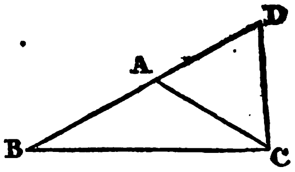
For the angle CAB being right, CBA is acute [Cor. 17. 1], therefore CB is greater than CA (by this prop).

Cor. 2.—If perpendiculars let fall from two points (C, F), on two right lines [AB, DE], be equal; and from the same points, right lines [CB, FE], be drawn to points [B, E], in those right lines, at unequal distances [AB, DE], from the incidences [A, D] of the perpendiculars, that [FE], which is drawn to the most distant point [E], is greater than the other [CB].

On DE, take DG equal to AB [3. 1], and draw FG; in the triangles CAB, FDG, CA, AB and the angle CAB, are severally equal to FD, DG and the angle FDG, therefore FG is equal to CB [4. 1]; and the external angle FGE of the triangle FDG is greater than the interior FDG [16. 1]; whence, FDG being a right angle, FGE is obtuse, and therefore FEG acute [Cor. 17. 1], and so FE greater than FG [19. 1], or its equal CB.

PROP. XX. THEOR.

Any two sides (as BA, AC) of a triangle (ABC), are greater than the remaining side (BC).

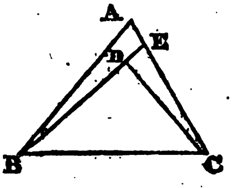


On either of the sides to be proved greater than the remaining one, as BA, produced, take AD equal to the other AC, and join CD.

The angles D and ACD at the base of the isosceles triangle ACD are equal [5. 1], whence BCD, being greater than ACD [Ax. 9], is greater than the angle D ; therefore, in the triangle BCD, the side BD, opposite the greater angle BCD, is greater than the side BC, opposite the less angle D [19. 1] ; but, because AD is equal to AC, BD is equal to BA and AC together [Ax. 2], therefore BA and AC together are greater than BC.

PROP. XXI. THEOR.

Two right lines (DB, DC), drawn from any point (D), within a triangle (ABC), to the extremes of any side (BC), are together less than the other sides (AB, AC) of the triangle, but contain a greater angle (BDC).

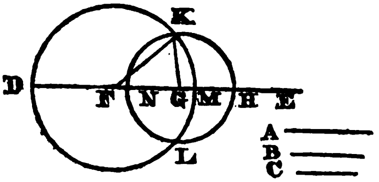


Produce BD to meet AC in E, and because the sides BA, AE of the triangle BAE, are greater than the remaining side BE (20. 1), adding to each EC, the right lines BA and AC, are greater than BE and EC [Ax. 4]; but the sides DE, EC of the triangle DEC, are greater than the remaining side DC (20. 1), therefore, adding to each BD, the right lines BE and EC are greater than BD and DC [Ax. 4]; but it has been proved, that BA and AC are greater than BE and EC, therefore BA and AC are also greater than BD and DC.

And, because the external angle BDC, of the triangle DEC, is greater than the internal remote angle BEC (16. 1), and the external angle BEC, of the triangle ABE, is greater than the internal remote angle A, the angle BDC is greater than A.

PROP. XXII. PROB.

Three right lines (A, B and C) being given, any two of which are greater than the remaining one, to make a triangle having its sides severally equal to these right lines.



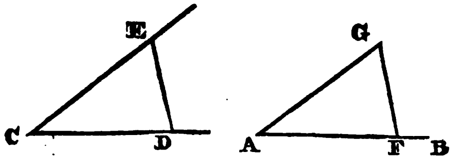
At any point D, put the right line DF equal to A (2. 1), on which produced take FG equal to B, and GH to C (3. 1); from the centre F, at the distance FD, describe the circle DKM, and from the centre G, at the distance GH, describe the circle NKH [Post. 3]; and, since A and C, or their equals FM, NG together, are greater than B or FG [Hyp.], taking NG from each, FM is greater than FN [Ax. 5]; and B and C, or their equals FG, GH together, or FH, is greater than A or FM [Hyp.]; whence, FM being greater than FN and less than FH, the point M is between the points N and H: in like manner, the point N may be shewn to be between the points D and M: whence, the circles DKM, NKH, being continued lines between the points D and M, and the points N and H, on each side of the right line DE [Def. 10], intersect each other on each side thereof, as in K and L: join FK, KG; the triangle FKG has its sides severally equal to the given right lines A, B and C.

For, because F is the centre of the circle DKM, FK is equal to FD [Def. 10], or its equal [Constr.] A; in like manner, GK may be proved equal to C; and FG is equal to B (Constr.)

Schol. The reasoning in this proposition to prove the intersection of the circles, is accommodated to the figure, but the same reasoning is manifestly applicable to all possible cases. (*See note on this proposition.*)

PROP. XXIII. PROB.

To a given right line (AB), at a given point therein (A), to make a rectilineal angle, equal to a given rectilineal angle (C).

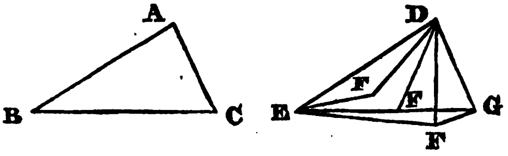


Take any points, D, E in the legs of the given angle C, join DE, and let the triangle FAG be made equilateral to the triangle DCE, so that its sides FA, AG meeting at the given point A may be severally equal to the legs DC, CE of the given angle C, and one of these sides AF may be taken on the given right line AB (22. 1). The angle A is equal to the given angle C.

For because the triangles FAG, DCE are mutually equilateral, the angles A, C, opposite the equal sides FG, DE, are equal (8. 1).

PROP. XXIV. THEOR.

If two triangles (ABC, DEF), have two sides (AB, AC) of the one, severally equal to two sides (DE, DF) of the other, but the angles (A and EDF) contained by these equal sides unequal, (the angle A being greater than the angle EDF,) the base (BC) of that triangle (ABC) which has the greater angle (A), is greater than the base (EF) of the other.



At the point D, with the right line DE, make the angle EDG equal to the angle A [23. 1], take DC equal to AC, and draw EG.

Because the angle EDG or A is greater than EDF, the right line DG falls without the triangle EDF, and the point F falls either below the right line EG, or on it, or above it.

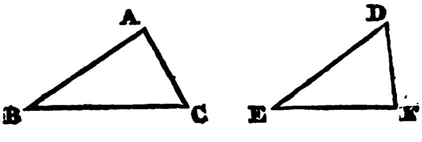
If the point F fall below EG, draw FG; and because DF, DG are equal, the angles DFG, DGF are also equal [5. 1]; whence the angle EFG being greater than DFG, and DGF than EGF (Ax. 9), the angle EFG is greater than EGF, therefore EG is greater than EF [19. 1.]; but, because ED, DG and the included angle EDG are severally equal to BA, AC and the included angle A, EG is equal to BC [4. 1]; therefore BC is also greater than EF.

If the point F fall on EG, the right line EG, or its equal BC, is, in that case, greater than EF [Ax. 9].

If F falls above EG, the right lines DG, GE together are greater than DF, FE together [21. 1]; taking from each the equals DG, DF, the residue EG, or its equal BC, is greater than the residue EF.

PROP. XXV. THEOR.

If two triangles (ABC, DEF), have two sides (AB, AC) of the one, severally equal to two sides (DE, DF) of the other, but the bases or third sides (BC, EF) unequal (BC being greater than EF); the angle (A) opposite the greater base (BC), s greater than that (D) opposite the less (EF).



For if the angle A be not greater than D, it is either equal to or less than it.

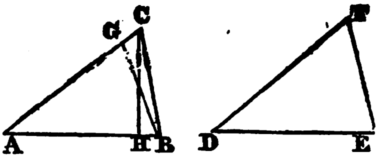
The angle A is not equal to D, for, if it were, the sides BA, AC being severally equal to ED, DF, the bases BC, EF would be equal [4. 1]; contrary to the hypothesis.

The angle A is not less than D, for, if it were, the sides BA, AC being severally equal to ED, DF, the base BC would be less than the base EF [24. 1]; which is again contrary the hypothesis.

Therefore the angle A, being neither equal to nor less than B, is greater than it.

PROP. XXVI. THEOR.

If two triangles (ABC, DEF), have two angles (A and ABC) of one, severally equal to two angles (D and E) of the other, and one side of one to one side of the other, namely either the sides (AB, DE) between the equal angles, or those (AC, DF) opposite to equal angles; the remaining sides are severally equal to each other, and the remaining angle (ACB) of the one, to the remaining angle (F) of the other.



Let first, AB be supposed equal to DE, then is AC equal to DF.

For if not, let one of them, as AC, be, if possible, the greater, and on AC take AG equal to DF, and draw BG.

In the triangles ABG, DEF, the sides AB, AG and the angle A, are severally equal to the sides DE, DF and the angle D [Constr. and Hyp.]; therefore the angles ABG and E are equal (4. 1); but the angles ABC and E are equal [Hyp.]; therefore the angles ABG, ABC, being each equal to E, are equal to each other [Ax. 1], part and whole, which is absurd [Ax. 9]; therefore the sides AC, DF are not unequal, they are therefore equal; whence AB being equal to DE, and the angle A to the angle D [Hyp.], BC is equal to EF, and the angle ACB to the angle F (4. 1).

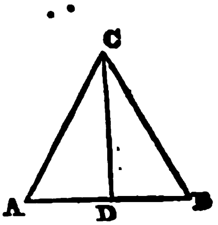
Let now AC, DF, opposite the equal angles, ABC and E, be supposed equal, then is AB equal to DE.

For if not, let one of them, as AB, be the greater, and on AB take AH equal to DE, and draw CH.

In the triangles AHC, DEF, the sides AH, AC and the angle A, are severally equal to the sides DE DF and the angle D [Constr. and Hyp.], therefore the angles AHC and E are equal [4. 1]; but the angles ABC and E are equal [Hyp.], therefore the angles AHC, ABC, being each equal to E, are equal to each other [Ax. 1], namely, the external angle AHC of the triangle CHB to the internal remote angle ABC, which is absurd [16. 1]; therefore AB, DE are not unequal, they are therefore equal; whence, AC being equal to DF, and the angle A to the angle D [Hyp.], BC is equal EF, and the angle ACB to the angle F [4. 1].

Cor.—A right line [CD], drawn from the vertex [C] of an isosceles triangle [ACB], bisecting the base [AB], is perpendicular to it; and, if it be perpendicular to the base, it bisects it.

Part 1.—Let CD bisect AB, it is perpendicular to it. In the triangles CAD, CBD, the sides CD, DA, and the base CA, are severally equal to CD, DB and the base CB Hyp.], therefore the angle CDA, CDB are equal [8. 1], and so CD perpendicular to AB [Def. 20].

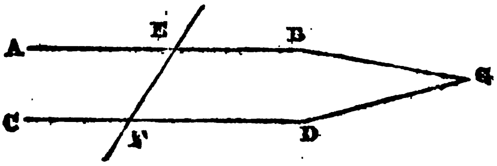


Part 2.—Let CD be perpendicular to AB, it bisects it. In the triangles ACD, BCD, the angles A and B are equal (5. 1), as are also the angles ADC, BDC [Hyp. and Def. 20], and CD opposite the equal angles A, B is common, therefore AD is equal to DB (26. 1).

Schol.—Only two equal right lines (CA, CB) can be drawn from the same point (C) to any right line (AB); since any two right lines, drawn from C to AB, on the same side of the perpendicular CD, are unequal [Cor. 2. 19. 1]. And these equal right lines form equal angles with that to which they are drawn (5. 1); and two right lines (CA, CB), drawn to any right line (AB), from a point (C) without it, and making equal angles (CAB, CBA) with it, are equal (6. 1).

PROP. XXVII. THEOR.

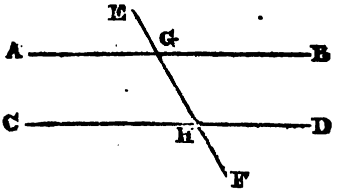
If a right line (EF), meeting two other right lines (AB, CD), make the alternate angles (AEF, EFD) equal; the right lines (AB, CD) so met, are parallel.



For, if AB and CD be not parallel, they may be so produced, as to meet either towards B, D or A, C [Def. 34.]; let them, if possible, being produced towards B,D, meet, as in G, and the external angle AEF, of the triangle EFG, is greater than the internal remote one EFG (16. 1), but it is also equal to it [Hyp.], which is absurd; therefore AB, CD do not meet towards B, D. In like manner it may be shewn, that they do not meet towards A, C: since therefore they do not meet towards either part, they are parallel.

PROP. XXVIII. THEOR.

If a right line (EF), cutting two other right lines (AB, CD), make an external angle (EGB), equal to the internal remote on the same side (GHD), or two internals on the same side (BGH, GHD) equal to two right angles; the right lines (AB, CD) so cut, are parallel.



First—Let EGB be equal to GHD; and because AGH is equal to EGB (15. 1), AGH is equal to GHD [Ax. 1], and they are alternate angles, therefore AB and CD are parallel (27. 1).

Let now the angles BGH, GHD be equal to two right angles; and because AGH, BGH are also equal to two right angles (13. 1), the angles BGH, GHD, together, are equal to AGH, BGH together; take away from each the common angle BGH, and the angle AGH is equal to GHD, and they are alternate angles, therefore AB and CD are parallel (27. 1).

PROP. XXIX. THEOR.

A right line (EF, see fig. to the preced. prop.), cutting two parallel right lines (AB, CD), makes the alternate angles equal, (AGH to GHD, and BGH to GHC); any external angle (as EGB), to the internal remote on the same side (GHD); and any two interior angles on the same side (as BGH, GHD, together equal to two right angles.

First.—The alternate angles, as AGH, GHD, are equal. For if not, let one of them, as AGH, be the greater; and the angles AGH, HGB being together equal to two right angles (13. 1), or, which is equal (13. 1. and Theor. at 11. 1), to CHG, GHD; taking from them the unequals AGH, GHD, the remainder CHG is greater than the remainder BGH [Ax. 5]; whence, AGH being greater than GHD [Hyp.], AGH, GHC together are greater than BGH, GHD together, therefore AB, CD may be so produced towards the part B, D, as to meet [Ax. 12], which is absurd [Hyp. and Def. 34]; therefore the angles AGH, GHD are not unequal, they are therefore equal. In like manner the angles BGH, GHC may be proved equal.

Secondly.—An exterior angle, as EGB, is equal to the interior remote on the same side GHD. For the angle EGB is equal to AGH [15. 1], and AGH is equal to the alternate GHD [part 1. of this], therefore EGB is equal to GHD [Ax. 1].

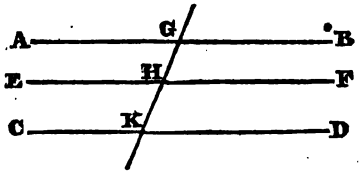
Thirdly.—Two interior angles on the same side, as BGH, GHD, are equal to two right angles. For the alternate angles AGH, GHD are equal [part 1. of this], adding to each BGH, the angles BGH, GHD together, are equal to AGH, BGH together [Ax. 2.], but AGH, BGH together are equal to two right angles [13. 1], therefore BGH, GHD together are equal to two right angles.

Theorem.—If a right line [EF], meeting two other right lines [AB, CD], make the interior angles on one side of it [BGH, GHD less than two right angles; these right lines may be so produced towards the part [B, D], on which the angles are less than two right angles, as to meet.

For, because the right line HG falls on AB, the angles AGH, HGB are together equal to two right angles [13. 1]; for the same reason, the angles CHG, GHD are together equal to two right angles; therefore the four angles AGH, HGB, CHG, GHD are together equal to four right angles; but the angles BGH, GHD are together less than two right angles [Hyp.], therefore the angles AGH, GHC are together greater than two right angles, and therefore greater than the angles BGH, GHD together, and of course the right lines AB, CD may be so produced towards the part B, D, as to meet [Ax. 12].

PROP. XXX. THEOR.

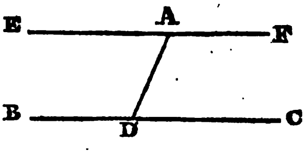
Two right lines (AB, CD), which are parallel to the same right line (EF), are parallel to each other.



Draw the right line GHK, cutting AB EF and CD in G, H and K. And, because AB, EF are parallel, and GK cuts them, the angle AGH is equal to the alternate GHF [29. 1]; and, because GK cuts the parallels EF, CD, the angle GHF is equal to the internal remote HKD [29. 1[; whence, the angles AGH, HKD, being each equal to GHF, are equal to each other [Ax. 1], and therefore AB and CD are parallel [27. 1].

PROP. XXXI. THEOR.

To a given right line (BC), through a given point (A) without it, to draw a parallel right line.

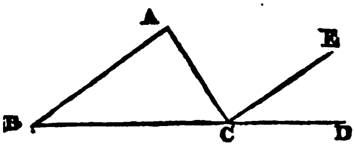


To any point D in BC, join AD, and at the point A, with the right line AD, make the angle DAE equal to ADC [23. 1], on contrary sides of the right line AD; the right line EAF is parallel to BC.

For the right line AD, meeting the right lines EF. BC, makes the alternate angles EAD, ADC equal; therefore EF is parallel to BC [27. 1].

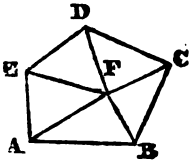
PROP. XXXII. THEOR.

If any side (BC) of a triangle (ABC) be produced, the exterior angle (ACD) is equal to the two interior remote ones (A and B). And the three interior angles (A, B and ACB), of every triangle (ABC), are equal to two right angles.



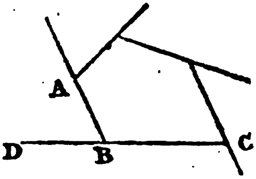
Through C draw CE parallel to BA [31. 1]. Because AC meets the parallels BA, CE, the alternate angles ACE, BAC are equal [29. 1]; and because BD meets the same parallels, the external angle ECD is equal to the interior remote on the same side ABC [29. 1]; therefore the angles ACE, ECD together, or the angle ACD, is equal to the two angles A and B together [Ax. 2].

To each of these equals add the angle ACB, and the two angles ACD and ACB are together equal to the three angles A, B and ACB [Ax. 2]; but the angles ACD and ACB are together equal to two right angles [13. 1], therefore the angles A, B and ACB are equal to two right angles.



Cor. 1.—All the interior angles (ABC, BCD, CDE, DEA and EAB) of any rectilineal figure (ABCDE), are equal to twice as many right angles, except four, as the figure has sides.

Take any point F within the figure, and draw FA, FB, FC, FD, FE; there are formed as many triangles as the figure has sides, all the angles of which are equal to twice as many right angles as the figure has sides [by this prop.]; but of these all the angles about the point F are equal to four right angles [Cor. 2. 15. 1]; therefore the remaining angles, which constitute the interior angles of the figure, are equal to twice as many right angles, except four, as the figure has sides.



Cor. 2.—All the exterior angles, of any rectilineal figure (AC), are together equal to four right angles.

For every exterior, as ABD and its adjacent interior ABC, are together equal to two right angles [13. 1]; therefore all the exterior and interior angles of the figure are together equal to twice as many right angles as the figure has sides; but the interior angles are equal to twice as many right angles, except four, as the figure has sides [by the preceding Cor.], therefore the exterior angles are equal to these four.

Cor. 3.—If from the vertex (C, see fig. 1, 2 and 3), of an isosceles triangle (ABC), a right line (CD) be drawn without the triangle, equal to one of its equal sides (AC or CB), the angle (ADB) formed at its other extreme (D), by right lines (DA, DB) drawn to the extremes (A, B) of the base, is equal to half the vertical angle (ACB).

Fig. 1.

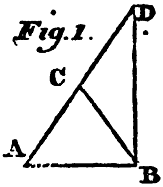


Fig. 2.

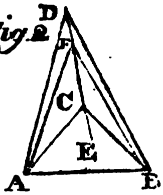
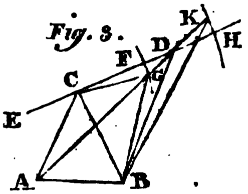


Fig. 3.



First, let one of the right lines, drawn from D to the extremes of the base, pass by the vertex C, as in fig. 1; and, since the triangles CBD is isosceles, the angles CBD, CDB are equal [5. 1], whence, the external angle ACB of the triangle BCD, being equal to the two internal remote angles CBD, CDB [by this prop.], is double to ADB.

Secondly, let the point C be within the triangle ADB, as in fig. 2. Draw DC, which produce beyond C, as to E; and, by case 1, the angle ACE is equal to the double of ADC, and BCE to the double of BDC, and therefore, the whole ACB to the double of the whole ADB [Ax. 2]:

Thirdly, let the point C be without the triangle ADB, as in fig. 3. Draw DC, which produce beyond C, as to E; the angle ECB is [by case 1,] equal to the double of EDB, and ECA to the double of EDA; therefore, taking the latter from the former, ACB is equal so the double of ADB [Ax. 3].

Cor. 4.—If the angle (ADB, see fig. 1. 2 and 3,] formed at any point (D), above the base (AB), of an isosceles triangle (ACB), by right lines (DA, DB) drawn to the extremes of the base. be equal to half the vertical angle (ACB); the right line (CD), joining that point to the vertex of the triangle, is equal to one of the equal sides of the triangle (AC or CB).

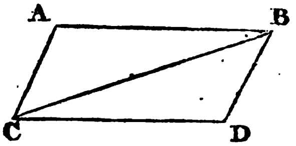
First, let CD be in the same right line with one of the equal sides AC, as in fig. 1 above; then, since ACB is equal to CBD and CDB [by this prop.], and CDB equal to the half of ACB [Hyp.], CBD is also the half of ACB; therefore the angles CBD, CDB are equal [Ax. 7], and therefore CD is equal to CB [6. 1].

Secondly, let the angle ACB be included within the angle ADB, as in fig. 2 above; CD is in this case also equal to CB; for, if not, it is either greater or less than it; let CD, if possible, be greater than CB, and take thereon CF equal to CB [3. 1], and join FA, FB; the angle AFB is equal to half the angle ACB [by preceding Cor.], but ADB is equal to half the angle ACB [Hyp.], therefore the angles AFB, ADB are equal [Ax. 7], which is absurd [21. 1], therefore CD is not greater than CB; in like manner it may be shewn, that CD is not less than CB; it is therefore equal to it.

Thirdly, let C be without the triangle ADB, as in fig. 3 above; CD is in this case also equal to CB; for, if not, it is either greater or less than it; and first, let CD, if possible, be greater than CB, and on it take CF equal to CB; from the centre C, at the distance CF, describe an arch of a circle FG [Post. 3], which, (because CF and CB are equal), being continued, would pass through B, whence, the points F, B of the circumference, being on contrary sides of AD, that circumference must meet AD between these points; let it meet AD in G, and join FA, CG, GB; and because CG is equal to CB [Def. 10], the angle AGB is half of ACB [by last Cor.], but ADB is also half of ACB [Hyp.], therefore the angles AGB, ADB are equal (Ax. 7), the external angle AGB of the triangle GBD to the internal remote GDB, which is absurd (16. 1); therefore CD is not greater than CB: Let now CD, if possible, be less than CB, and, on CD produced, take CH equal to CB; from the centre C, at the distance CH, describe an arch of a circle, meeting AD produced in K, and join KB; and because a right line joining the points C, K is equal to CH (Def. 10), or its equal (Constr.) CB, the angle AKB is half the angle ACB [by preced. Cor.], but the angle ADB is also half of ACB (Hyp.), therefore the angles ADB, AKB are equal (Ax. 7), the external angle ADB of the triangle DBK to the internal remote DKB, which is absurd (16. 1); therefore CD is not less than CB; and it is above shewn not to be greater than CB; since therefore CD is neither greater nor less than CB, it is equal to it.

PROP. XXXIII. PROB.

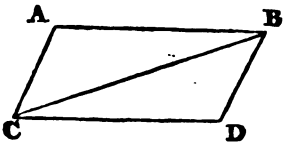
Right lines (AC, BD), joining the adjacent extremes of equal and parallel right lines (AB, CD), are themselves equal and parallel.



Join CB, and in the triangles ABC, BCD, the sides AB, CD, are equal [Hyp.], BC common, and, because AB and CD are parallel [Hyp.], the alternate angles ABC, BCD are equal [29. 1], therefore [by 4. 1,] AC is equal to BD, and the angle ACB to CBD; but these are alternate angles, formed by the right line BC meeting the right lines AC, BD; therefore AC and BD are parallel [27. 1].

PROP. XXXIV. THEOR.

The opposite sides (AB, CD, and AC, BD), and opposite angles (A, D, and ABD, ACD), of a parallelogram (AD), are equal, and a diagonal, or right line, joining two of its opposite angles (as CB), bisects it.



In the triangles ABC, BCD, the angles ABC, BCD are equal, as also the angles ACB, CBD, being alternate angles, formed by BC meeting the parallels AB, CD, and AC, BD [29. 1]; but the side CB, between the equal angles, is common to both these triangles, therefore AB is equal to CD, AC to BD, and the angle A to D [26. 1]; and, because AB, CD are parallel, the angles A and ACD are equal to two right angles [29. 1], or, which is equal [29. 1 and Theor. at 11, 1], to the angles D and ABD; taking from each, the equal angles A, D, the remaining angles ACD, ABD are equal [Ax. 3].

And, since AB, BC and the included angle ABC, are severally equal to CD, CB and the included angle BCD, the triangles ABC, BCD are equal [4. 1]; therefore CB bisects the parallelogram AD.

Cor. 1.—If one angle of a parallelogram be right, the rest are right.

For either adjacent angle is right, because it and a right angle are equal to two right angles [29. 1]; and the opposite angles are right, because equal to these right angles [by this prop.]

Cor. 2.—Two parallelograms, which have one angle of one equal to one angle of another, are mutually equiangular.

For the angles, which are opposite these equal angles, are equal to them [34. 1], and therefore to each other; and the angles which are adjacent to them, because with them they are equal to two right angles [29. 1], are also equal.

Cor. 3—Two parallelograms, which are mutually equiangular and equilateral, are equal.

For, drawing a diagonal in each, subtending equal angles, either of the two triangles into which one of the parrallelograms is divided, is equal to either of those into which the other is divided [4. 1], therefore the parallelograms, which are double to these triangles [34. 1], are equal [Ax. 6].

Cor. 4.—Every quadrilateral figure [ABDC], whose opposite sides are equal, [AB to CD, and AC to BD,] is a parallelogram.

Join CB, and, in the triangles ABC, DCB, the sides CA, AB are severally equal to BD, DC [Hyp.], and BC is common, therefore the angles A, D are equal [8. 1]; whence also, the angle ABC is equal to BCD, and ACB to CBD [4. 1], therefore AB is parallel to CD, and AC to BD [27. 1], and ABDC is a parallelogram [Def. 35].

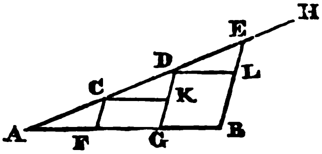
Cor. 5.—Every quadrilateral figure [ABDC], whose opposite angles are equal, [A to D, and ABD to ACD,] is a parallelogram.

Because the angle A is equal to D, and ABD to ACD [Hyp.], the four angles A, ABD, D and ACD are double to the two angles A and ABD; but the four angles A, ABD, D and ACD, of the quadrilateral figure ABCD, are equal to four right angles [Cor. 1. 32. 1]; therefore the angles A and ABD are equal to two right angles, and therefore AC and BD are parallel [28. 1]. In like manner it may be proved, that AB and CD are parallel; of course ABDC is a parallelogram [Def. 35].

Cor. 6.—Two finite right lines [AB, AC], meeting in an angle, being given, to make a parallelogram, whereof they are sides.

Through B draw BD parallel to AC, and through C, CD parallel to AB [31. 1]; and, having joined CB, because the two angles ABC, ACB of the triangle ABC, are less than two right angles [17. 1], the two angles BCD, CBD, being alternates to them, and therefore severally equal to them [29. 1], are also less than two right angles; therefore BD, CD may be so produced as to meet [Theor. at 29. 1]; let them meet, as in D; the quadrilateral ABDC is a parallelogram, whereof the given right lines AB, AC are sides [Def. 35].

Cor. 7.—To divide a given right line [AB], into any required number of equal parts.

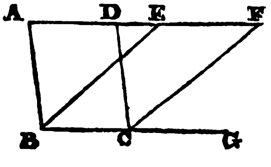


Draw AH, making any angle whatever with AB, and take thereon as many equal parts AC, CD, DE, as those into which AB is required to be divided [3. 1], draw EB, and parallel thereto CF, DG [31. 1]; AB is divided as required in F and G.

For CK, DL, being drawn parallel to AB, are parallel to each other [30. 1]; and, since AH meets the parallels CK, DL, the angle EDL is equal to the internal remote on the same side DCK [29. 1]; and, since AH meets the parallels DG, EB, the angle CDK is equal to the internal remote on the same side DEL [29. 1]; therefore, in the triangles CDK, DEL, the angles DCK, CDK are severally equal to EDL, DEL, and the sides CD, DE between the equal angles are equal [Constr.], therefore CK is equal to DL [26. 1]. In like manner, AF may be proved equal to CK; but, because of the parallelogram CG, the right line FG, is equal to CK [34. 1]; whence AF, FG, being each equal to CK, are equal to each other [Ax. 1]. In like manner FG, GB may be proved equal, and so the parts AF, FG, GB, into which the given right line is divided, are equal.

PROP. XXXV. THEOR.

Parallelograms (BD and BF), on the same base (BC), and between the same parallels (BC, AF), are equal.



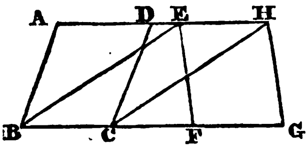
Produce BC, as to G. The external angle DCG is equal to the internal remote angle ABC [29. 1]; taking from these equals, the external FCG and internal EBC, which are also equal [29. 1], the remainders DCF, ABE are equal [Ax. 3];

whence the triangles DCF, ABE, having also DC, CF, severally equal to AB, BE [34. 1], are equal [4. 1]; but the triangle DCF being taken from the quadrangle ABCF, there remains the parallelogram BD; and the triangle ABE being taken from the same quadrangle, there remains the parallelogram BF; the parallelograms BD, BF are therefore equal [Ax. 3].

Scholium.—The demonstration is evidently the same, though the point E should coincide with D, or be between the points A and D.

PROP. XXXVI. THEOR.

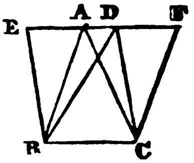
Parallelograms (BD, FH), on equal bases (BC, FG), and between the same parallels (BG, AH), are equal.



Join BE, CH, and because BC is equal to FG [Hyp.], and EH to FG (34. 1), BC is equal to EH (Ax. 1); and BC, EH are parallel [Hyp.], therefore BE, CH, which join them, are parallel (33. 1), and BH is a parallelogram (Def. 35), and is equal to the parallelogram BD, being on the same base BC, and between the same parallels BC, AH (35. 1); and the parallelogram BH is also equal to FH, being on the same base EH, and between the same parallels BG, EH (35. 1); whence, the parallelograms BD and FH, being each equal to the parallelogram BH, are equal to each other.

PROP. XXXVII. THEOR.

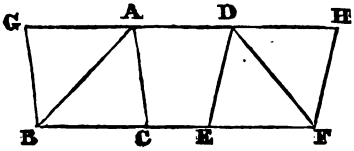
Triangles (ABC, DBC), on the same base (BC), and between the same parallels (BC, AD), are equal.



Through B, draw BE parallel to CD, and through C, CF parallel to BA [31. 1]; meeting AD produced in E and F; the parallelograms EC, BF are equal, being on the same base BC, and between the same parallels BC, EF [35. 1]; but the triangles DBC, ABC are the halves of these parallelograms, because the diagonals BD, AC bisect them [34. 1]; therefore these triangles are equal [Ax. 7].

PROP. XXXVIII. THEOR.

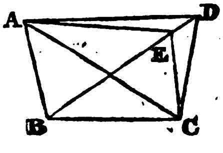
Triangles (ABC, DEF), on equal bases (BC, EF), and between the same parallels (BF, AD), are equal.



Through B, draw BG parallel to CA, and through F, FH parallel to ED [31. 1], meeting AD produced in G and H; the parallelograms GC, EH are equal, being on equal bases BC, EF, and between the same parallels BF, GH, [36. 1]; whence, the triangles ABC, DEF, being the halves of these parallelograms [34. 1], are also equal [Ax. 7].

PROP. XXXIX. THEOR.

Equal triangles (ABC, DBC), on the same base (BC), and on the same side of it, are between the same parallels.

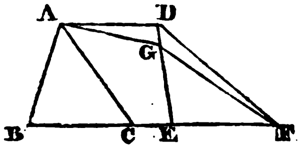


Join AD, which is parallel to BC; for, if not, through A, draw AE parallel to BC [31. 1], meeting either side BD of the triangle BDC, in a point E, different from the vertex, and join EC.

The triangles ABC, EBC, being on the same base BC, and between the same parallels BC, AE, are equal [37. 1]; but the triangle DBC is equal to ABC [Hyp.]; therefore the triangles EBC, DBC are equal [Ax. 1], part and whole, which is absurd; therefore AE is not parallel to BC. In like manner it may be shewn, that no other right line drawn through A, except AD, is parallel to BC; therefore AD is parallel to BC.

PROP. XL. THEOR.

Equal triangles (ABC, DEF), on equal bases in the same right line (BC, EF), and towards the same part, are between the same parallels.

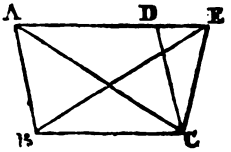


Join AD which is parallel to BF; for, if not, through A, draw AG parallel to BF [31. 1], cutting either side DE of the triangle DEF, in a point G different from the vertex, and join GF.

The triangles ABC, GEF, being on equal bases BC, EF, and between the same parallels BF, AG, are equal [38. 1]; but the triangle DEF is equal to ABC [Hyp.]; therefore the triangles DEF, GEF are equal [Ax. 1], whole and part, which is absurd; therefore AG is not parallel to BF. In like manner it may be shewn, that no other right line drawn through A, except AD, is parallel to BF; therefore AD is parallel to BF.

PROP. XLI. THEOR.

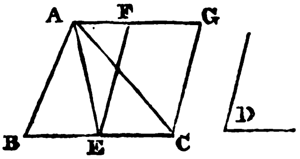
If a parallelogram (BD), and triangle (BEC), be on the same base (BC), and between the same parallels (BC, AE), the parallelogram is double to the triangle.



Join CA. The triangle BEC is equal to BAC, being on the same base BC, and between the same parallels BC, AE, [37. 1]; but the parallelogram BD, being bisected by the diagonal AC [34. 1], is double the triangle BAC; therefore the parallelogram BD is also double the triangle BEC.

PROP. XLII. PROB.

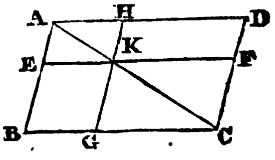
To constitute a parallelogram, equal to a given triangle (ABC), and having an angle, equal to a given rectilineal angle (D).



Through A, draw AG parallel to BC [31. 1], bisect BC in E (10. 1), and at the point E, with the right line EC, make the angle CEF equal to the given angle D (23. 1); through C, draw CG parallel to EF (31. 1), meeting AG in G, and join AE; ECGF is a parallelogram (Def. 35), and double the triangle AEC (41. 1); and since the triangles ABE, AEC are equal, being on equal bases BE. EC, and between the same parallels BC, AG (38. 1), the triangle ABC is double the triangle AEC; whence, the parallelogram EG and triangle ABC, being each of them double the triangle AEC, are equal (Ax. 6); and one angle FEC of the parallelogram is equal to the given angle D (Constr.). There is therefore constituted a parallelogram EG, equal to the given triangle ABC, and having an angle FEC, equal to the given angle D, as was required.

PROP. XLIII. THEOR.

In any parallelogram (BD), the complements (BK, KD), of the parallelograms (EH, GF), which are about the diagonal (AC), are equal.

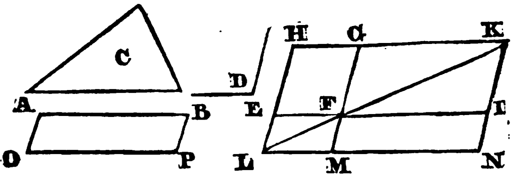


The parallelograms EH, GF, about the diagonal AC, are formed, by drawing through any point K of that diagonal, the right lines EF, HG parallel to BC, AB; and BK, KD are their complements to the whole, which are equal.

For, because AC bisects the parallelogram BD (34. 1), the triangles ABC, ADC are equal; and, because the same diagonal bisects the parallelograms EH, GF (34. 1), the triangles AEK, KGC are severally equal to the triangles AHK, KFC; therefore the triangles AEK, KGC together, are equal to AHK, KFC together (Ax. 2), which being taken from the equal triangles ABC, ADC, the remaining complements BK, KD are equal (Ax. 3).

PROP. XLIV. PROB.

To a given right line (AB), to apply a parallelogram, equal to a given triangle (C), and having an angle, equal to a given rectilineal angle (D).

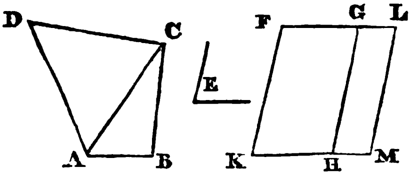


Make the parallelogram EFGH equal to C, having an angle EFG equal to D (42. 1); in EF produced, take FI equal to AB (3. 1), complete the parallelogram GI (Cor. 6. 34. 1), and join FK; because the angle FGK is equal to the internal remote angle EHG (29. 1), and the angles FGK, FKG are together less than two right angles (17. 1), the angles EHK, FKH are also together less than two right angles, therefore HE, KF may be so produced towards E, F, as to meet (Theor. at 29. 1); let them, being produced, meet, as in L, and through L, draw LN parallel to EI, meeting GF, KI produced in M and N; make the angle BAO equal to IFM (23. 1), AO to FM, and complete the parallelogram AOPB (Cor. 6. 34. 1)

In the parallelogram HN, the complements HF, FN are equal (43. 1), but the parallelogram HF is equal to C (Constr.), therefore the parallelogram FN is equal to C (Ax. 1); and the parallelogram AP is equal to FN (Cor. 3. 34. 1), therefore the parallelogram AP is also equal to C (Ax. 1), and it is applied to the given right line AB, having an angle A, equal to the angle MFI (Constr.), and therefore [15. 1], to EFG, and therefore [Constr.], to D.

PROP. XLV. PROB.

To make a parallelogram, equal to a given right-lined figure (ABCD), and having an angle, equal to a given rectilineal angle (E).

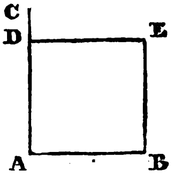


Join AC, and make the parallelogram KG, equal to the triangle ACD, having the angle K, equal to the angle E (42. 1); and to the right line GH, apply the parallelogram GM, equal to the triangle ABC, having the angle GHM equal to E (44. 1); the angles GHM and K, being each equal to the angle E (Constr.), are equal to each other, adding to each the angle GHK, the angles GHM, GHK together, are equal to K and GHK together (Ax. 2); but the angles K and GHK together, are equal to two right angles (29. 1), therefore the angles GHM, GHK together, are equal to two right angles, and therefore the right lines KH, HM make one right line (14. 1); and, since the angles HGF, HGL are severally equal to their alternate angles GHM, GHK (29. 1), the angles HGF, HGL together, are also equal to two right angles, and therefore the right lines FG, GL make one right line (14. 1); and FK and LM, being each of them parallel to GH, are parallel to each other (30. 1); and FL is parallel to KM; therefore FM is a parallelogram, equal to the given right-lined figure ABCD (Constr. and Ax. 2), and having an angle K equal to E (Constr.)

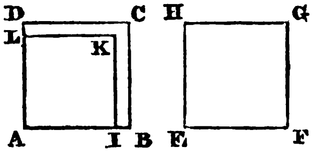
Cor.—Hence it appears, how there may be applied, to a given right line FK, a parallelogram, equal to a given rectilineal figure ABCD, and having an angle, equal to a given rectilineal one E; namely, by applying to FK, a parallelogram, equal to the triangle ACD, having an angle, equal to the given one (44. 1), and then proceeding, as in this proposition.

PROP. XLVI. PROB.

On a given right line (AB), to constitute a square.

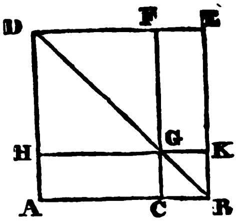


From the point A, draw AC at right angles to AB (11. 1), on which take AD equal to AB (3. 1), and complete the parallelogram AE (Cor. 6. 34. 1). Because AB, AD are equal (Constr.), and the sides DE, EB opposite to them, severally equal to AB, AD (34. 1), the four sides AB, AD, DE, EB are equal to each other, and the parallelogram AE is equilateral: it is also right angled, for the angle A being right (Constr.), its other angles are right (Cor., 1. 34. 1)); therefore ABED is a square (Def. 36), and is constituted on the given right line AB.



Cor. 1.—Equal squares (ABCD, EFGH), have equal sides, (as AB, EF).

For if AB and EF be not equal, let one of them, as AB, be the greater, and take thereon AI equal to EF, on which make the square AIKL (by this prop.); and because AI, AL are equal (Constr. and Def. 36), as also AB and AD (Hyp. and Def. 36), and AI is less than AB, AL is less than AD, and the square AIKL contained within the square ABCD; but the squares AIKL, EFGH, having equal sides, are equal (Cor. 3. 34. 1), and the squares ABCD, EFGH are also equal (Hyp.) whence, the squares AIKL and ABCD, being each equal to EFGH, are equal to each other (Ax. 1), part and whole, which is absurd (Ax. 9); therefore AB is not greater than EF. In like manner, it may be shewn, that AB is not less than EF. Therefore AB, being neither greater nor less than EF, is equal to it.

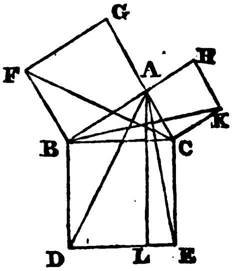


Cor. 2.—Parallelograms (CK, HF), about a diagonal (DB) of a square (AE), are squares.

Because all the angles of the triangle ABD are equal to two right angles (32. 1), and the angle A a right angle (Hyp. and Def. 36), the angles ABD, ADB are together equal to a right angle, and being, because of the equal sides AB, AD, equal to each other (5. 1), either of them, as ABD, is half a right angle; and, in the triangle CBG, the angle BCG is equal to the internal remote on the same side A (29. 1), and therefore a right angle, and CBG has been shewn to be half a right angle, therefore CGB is also half a right angle; whence, the angles CBG, CGB being equal, CB is equal to CG (6. 1); and, in the parallelogram CK, the sides opposite to these are severally equal to them (34. 1); therefore the parallelogram CK is equilateral; and because its angle at B is right (Hyp. and Def. 36), right angled (Cor. 1. 34. 1); it is therefore a square (Def. 36). In like manner it may be demonstrated, that HF is a square.

PROP. XLVII. THEOR.

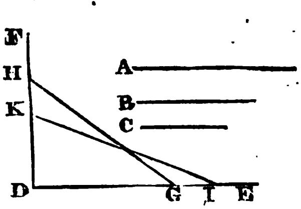
In every right angled triangle (ABC), the square of the side (BC), opposite the right angle (BAC), is equal to the squares of the other sides (AB and AC).



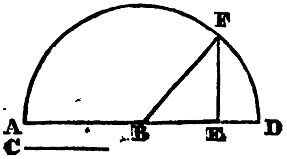
On the sides BC, BA, AC, describe the squares BE, FA, AK (46. 1), through A, draw AL parallel to BD or CE (31. 1), and join AD, FC.

Because the angles BAC, BAG are right angles (Hyp. and Def. 36), and therefore together equal to two right angles, the right lines GA, AC make one right line (14, 1); and because the angles FBA, DBC, being each of them right angles (Def. 36), are equal (Theor. at 11. 1), adding to each the angle ABC, the angles FBC, ABD are equal (Ax. 2), and the sides FB, BC are severally equal to the sides AB, BD (Def. 36), therefore the triangles FBC, ABD are equal (4. 1); but the parallelogram FA is double the triangle FBC, being on the same base FB, and between the same parallels FB, GC (41. 1); and the parallelogram BL is double the triangle ABD, being on the same base BD, and between the same parallels BD, AL (41. 1); whence, the parallelograms FA, BL, being double the equal triangles FBC, ABD, are equal (Ax. 6). In like manner, joining BK, AE, the parallelograms AK, CL may be proved equal; therefore the square BE of BC is equal to the squares FA, AK of the other sides AB, AC (Ax. 2).

Cor. 1.—Any number of squares being given, to find one equal to them all.



Let the right lines A, B and C be sides of the given squares; draw the right line DE, and, perpendicular thereto, from any point D therein, DF (11. 1); on DE, take DG equal to A, and on DF, DH equal to B (3. 1); join HG, and on DE, take DI equal to HG, and on DF, DK equal to C (3. 1); join KI, the square of which is equal to the squares of A, B and C; for it is equal to the squares of DI, DK (47. 1), and the square of DI, or its equal HG, is equal to the squares of DG, DH (47. 1); therefore the square of KI is equal to the squares of DG, DH and DK, or of their equals A, B and C.



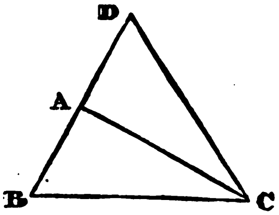
Cor. 2.—Two unequal right lines (AB and C) being given, to find a right line, whose square is equal to the excess of the square of the greater (AB), above that of the less (C).

From the centre B, at the distance BA, describe the semicircle AFD (Post. 3), meeting AB produced in D; BD and BA are equal (Def. 10), and therefore BD greater than C; on BD take BE equal to C (3. 1), and from the point E, draw EF perpendicular to BD (11. 1), meeting the arch AFD in F; EF is the right line required.

Join BF. The square of BF, or of its equal BA, is equal to the squares of BE and EF (47. 1), therefore the square of EF is equal to the excess of the square of AB above that of BE or C.

PROP. XLVIII. THEOR.

If the square of one side (BC), of a triangle (ABC), be equal to the squares of the other two sides (AB, AC); the angle (BAC) contained by these two sides, is a right angle.



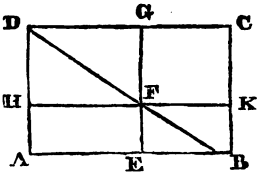
From the point A, draw AD, perpendicular to one of the sides AC containing the right angle, and equal to the other AB, and join DC.

Because DA is equal to AB, the square of DA is equal to the square of AB (Cor. 3. 34. 1), adding to each the square of AC, the squares of AD and AC are equal to the squares of BA and AC; but the square of DC is equal to the squares of DA and AC (47. 1), and the square of BC is equal to the squares of BA and AC (Hyp.), therefore the squares of DC and BC are equal, and therefore the right line DC is equal to BC (Cor. 1. 46. 1); whence, the right lines DA, AC being severally equal to BA, AC, the angles DAC, BAC are equal (8. 1); but the angle DAC is a right angle (Constr.), therefore the angle BAC is a right angle.

BOOK II.

DEFINITIONS.

1. A *Rectangle,* is a right angled parallelogram, (as ABCD,) and is said to be contained under any two of the right lines (AB, AD), which make one of its right angles.



The rectangle under two right lines, is that contained by these right lines, or by right lines equal to them, which is equal by Cor. 3. 34. 1.

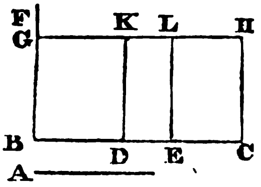
When a rectangle is denoted by three letters, the middle one is an extreme of both the right lines containing it, the other two letters being at the other extremes of these right lines; thus, the rectangle DAB, denotes the rectangle contained by the right lines DA, AB.

2. In any parallelogram, either of the parallelograms (HG or EK), which are about a diagonal, together with the two complements (AF and FC), is called a *Gnomon.*

A *gnomon* is generally denoted by three letters, which are at the opposite angles of the parallelograms which make the gnomon; thus, the gnomon, made by the parallelogram EK with the complements, is denoted by the leters AKG or HEC.

PROPOSITION I. THEOREM.

If there be two right lines (A and BC), whereof one (BC) is divided into any number of parts (BD, DE, EC); the rectangle under these right lines, is equal to the rectangles under the undivided line (A), and all the parts of the divided one (BD, DE and EC).

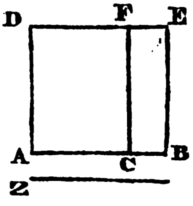


From B, draw BF perpendicular to BC (11. 1), on which take BG equal to A (3. 1), through G, draw GH parallel to BC, and through D, E, C, draw DK, EL, CH parallel to BG (31. 1), meeting GH in K, L, H.

The rectangle BH is equal to all its parts the rectangles BK, DL, EH (Cor. Ax. 8), but, because BG is equal to A (Constr.), the rectangle BH is equal to the rectangle under BC and A (Cor. 3. 34. 1), and because BG is equal to A (Constr.), and DK, EL each equal to BG (34. 1), and therefore to A (Constr. and Ax. 1), the parallelograms BK, DL, EH are equal to those under BD and A, DE and A, and EC and A.

PROP. II. THEOR.

If a right line (AB), be divided into any two parts (AC, CB); the square of the whole (AB), is equal to the rectangles (BAC, ABC), under the whole (AB), and each of the parts (AC, CB)



On AB describe the square AE (46. 1), and through C, draw CF parallel to AD [31. 1].

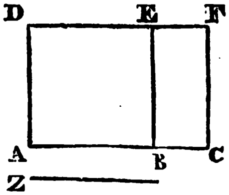
The square AE of AB, is equal to the rectangles AF, CE [Cor. Ax. 8]; but the rectangle AF is equal to the rectangle under AB, AC, because AD is equal to AB; and the rectangle CE is equal to the rectangle under AB, CB, because BE is equal to AB.

Otherwise.

Take the right line Z equal to AB [3. 1]. The rectangle under Z and AB, or, which is equal [Cor. 3. 34. 1], the square of AB, is equal to the rectangles under Z and AC, and under Z and CB [1. 2]; or, which is equal [Cor. 3. 34. 1], to the rectangles under AB, AC, and under AB, CB.

PROP. III. THEOR.

If a right line (AC), be divided into any two parts (AB, BC); the rectangle (CAB), under the whole (AC), and one of the parts (AB), is equal to the rectangle (ABC) under the parts, with the square of the first mentioned part (AB).



On AB describe the square AE [46. 1], and through C draw CF parallel to BE, meeting DE produced in F.

The rectangle AF, is equal to the square AE, with the rectangle BF; but AF is the rectangle under AC, AB, because AD is equal to AB [Constr. and Def. 36. 1], AE is the square of AB (Constr.), and BF is the rectangle under AB, BC, because BE is equal to AB.

Otherwise.

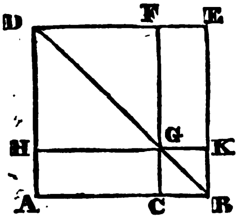
Take Z equal to AB [3. 1]. The rectangle under Z and AC, is equal to the rectangles under Z and AB, and under Z and BC [1. 2]; but, because Z is equal to AB [Constr.], the rectangle under Z and AC, is the rectangle CAB, the rectangle under Z and AB, the square of AB, and the rectangle under Z and BC, the rectangle ABC.

Scholium.—This proposition and the preceding, may be virtually comprised in one, namely,

The square of the greater, of two unequal right lines, is greater than the rectangle under the greater and their difference, and the square of either of two right lines, is less than the rectangle under it and their sum, by the rectangle under the same right lines; as is manifest, by supposing, in both these propositions, the right lines to be AB, BC.

PROP. IV. THEOR.

If a right line (AB), be cut into any two parts (AC, CB), the square of the whole (AB), is equal to the squares of the parts (AC, CB), with double the rectangle (ACB) under the parts.



On AB describe the square ABED [46. 1], draw BD, through C, draw CF parallel to AD, meeting BD in G, and through G, draw HK parallel to AB.

The square AE, is equal to CK, HF, with the rectangles AG, GE.

But CK is the square of CB [Constr. and Cor. 2. 46. 1]; and, because HG is equal to AC [34. 1], HF is the square of AC [Constr. and Cor. 2. 46. 1]; and AG is the rectangle under the parts AC, CB, because CG is equal to CB [Def. 36. 1]; whence, the complements AG, GE being equal [43. 1], GE is also the rectangle under AC, CB.

Otherwise.

The square of AB is equal to the rectangles BAC and ABC [2. 2]; but the rectangle BAC is equal to the rectangle ACB with the square of AC [3. 2], and the rectangle ABC is equal to the rectangle ACB, with the square of CB [3. 2], therefore the square of AB, which is equal to the rectangles BAC and

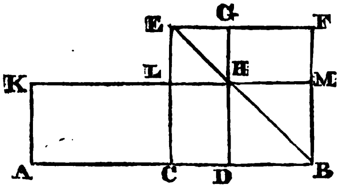
ABC, is equal to the squares of AC and CB, with double the rectangle ACB.

Cor.—The square of a right line, is four-fold the square of its half.

For, a right line being bisected, the rectangle under the parts, is equal to the square of the half line.

PROP. V. THEOR.

If a right line (AB), be divided into two equal parts (AC, CB), and two unequal parts (AD, DB); the rectangle under the unequal parts (AD, DB), with the square of the intermediate part, or part between the points of section (CD), is equal to the square of the half line (CB).



On CB, describe the square CBFE [46. 1], draw BE, and through D, DG parallel to BF, meeting BE in H; through H draw KLM parallel to AB, which let a right line, drawn through A, parallel to CE, meet in K.

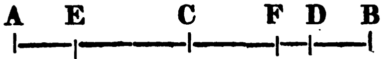
Because AC, CB are equal [Hyp.], the rectangles AL, CM are equal [36. 1]; and the rectangles CH, HF, being complements, are equal [43. 1]; therefore the rectangle AH is equal to the gnomon CMG [Ax. 2]; adding to each the square LG, the rectangle AH, with the square LG, is equal to the square CF; but the rectangle AH is the rectangle under AD, DB, because DH is equal to DB [Cor. 2. 46. 1. and Def. 36. 1], LG is the square of LH [Cor. 2. 46. 1], or its equal [34. 1] CD, and CF is the square of CB.

Otherwise.

The rectangle ADB is equal to the rectangles under AC, DB and under CD, DB [1 .2]; but the rectangle under AC, DB is, because of the equals AC, CB, equal to the rectangle CBD, or, which is equal [3. 2], to the rectangle CDB, with the square of DB; therefore the rectangle ADB is equal to double the rectangle CDB, with the square of DB; adding to each the square of CD, the rectangle under AD, DB with the square of CD is equal to double the rectangle CDB with the squares of CD, DB, or, which is equal [4. 2], to the square of CB.

Cor. 1.—Hence, the rectangle under the parts of any right line divided into two parts, is greatest, when the right line is bisected, as in C; and, the nearer the point of division is to the middle of the line, the greater the rectangle; the difference between it and the square of the half line, being [by this prop.] the square of the intermediate part CD.

Cor. 2.—If from the extremes [A, B], of any right line [AB], equal parts [AE, DB] be taken, and any point [F] be taken between them; the rectangle [AFB], under the segments [AF, FB] of the whole line [AB] between its extremes and the last assumed point, is equal to a rectangle [ADB], under the segments [AD, DB] of the same whole line between its extremes and one of the first assumed points, with the rectangle [EFD] under the segments between the last assumed point [F], and those [E, D], which are equidistant from the extremes of the whole.

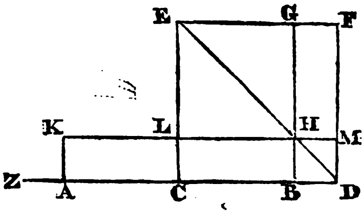


Bisect AB in C.

The rectangle AFB with the square of CF, is equal to the square of CB [5. 2]; or, which is equal [by the same], to the rectangle ADB with the square of CD; or, the square of CD being equal to the rectangle EFD with the square of CF [5. 2], to the rectangles ADB, EFD and the square of CF; taking from each the common square of CF, there remains the rectangle AFB, equal to the rectangles ADB, EFD.

PROP. VI. THEOR.

If a right line (AB) be bisected, and another right line (BD) be added in continuation; the rectangle (ADB) under the whole compound (AD) and the part added (BD), with the square of the half line (CB), is equal to the square of the compound of the half line and part added (CD).



On CD describe the square CDFE [46. 1], draw DE, through B, draw BG parallel to DF, meeting ED in H; and through H, draw MK parallel to AD, meeting a right line, drawn through A parallel to CE, in K.

Because AC, CB are equal, the rectangles AL, CH are equal [36. 1], but CH, HF. being complements, are equal [43. 1], therefore the rectangles AL, HF are equal [Ax. 1. 1]; adding to each the rectangle CM, the rectangle AM is equal to the gnomon CMG, and adding to each of these LG, the rectangles AM, LG are equal to the square CF of CD.

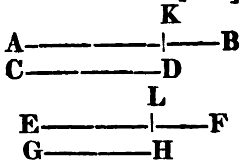
But AM is the rectangle under AD and BD, for DM is equal to DB [Cor. 2. 46. 1. and Def. 36. 1]; and LG is the square of LH [Cor. 2. 46. 1], and therefore of CB, which is equal to LH [34. 1].

Otherwise.

On the given right line DA produced beyond A, take AZ equal to BD [3. 1], adding to each AB, ZB is equal to AD; and ZD is divided equally in C, and unequally in B, therefore the rectangle ZBD with the square of CB, is equal to the square of CD [5. 2], and therefore, the rectangle ZBD being equal to the rectangle ADB, because ZA is equal to BD and ZB to AD, the rectangle ADB with the square of CB, is equal to the square of CD.

Scholium.—A like observation, as is made in the scholium to prop. 3 of this book, is applicable to this proposition and the preceding, which may be virtually comprised in one proposition, namely, "*The difference of the squares of two right lines, is equal to the rectangle under their sum and difference;*" as is manifest, by supposing, in both propositions, the right lines to be AC, CD.

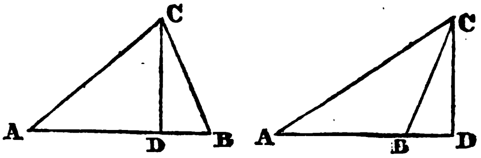
Theorem.—If the excess of the first [AB] of four magnitudes [AB, CD, EF, GH] above the second [CD], be equal to the excess of the third [EF] above the fourth [GH], the first [AB] being greater than the third [EF]; the excess of the first [AB], above the third [EF], is equal to the excess of the second [CD] above the fourth (GH).



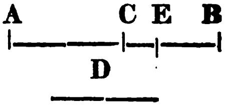
Let AK be a part taken on AB equal to CD, and EL a part on EF equal to GH; and, KB, LF are the excesses of AB above CD, and of EF above GH, and are equal (Hyp.); whence the excess of AB above EF, being equal to the excess of AB above EL and LF together, or above their equals GH and KB together; taking from each KB, the excess of AB above EF, is equal to the excess of AK, or its equal CD, above GH.

Corollaries to the two preceding propositions.

Cor. 1.—If a perpendicular (CD) be let fall on the base (AB) of a triangle, from the opposite angle (ACB); the rectangle under the sum and difference of the sides [AC, CB), is equal to the rectangle under the sum and difference of the segments (AD, DB) of the base, intercepted between its extremes (A, B), and the perpendicular (CD).

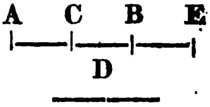


Let AC, see both figures to this cor., be that side, which is not the less of the two; and the difference of the squares of AC and AD, and of BC and BD are equal, being each equal to the square of CD (47. 1); therefore by (Theor. at this prop.) the difference of the squares of AC and CB, or, which is equal (Schol. to this prop.), the rectangle under the sum and difference of AC and CB, is equal to the difference of the squares of AD and BD, or, which is equal (Schol. to this prop.), the rectangle under the sum and difference of AD and DB.



Cor. 2.—To divide a given right line (AB) so into two parts, that the rectangle under the parts, may be equal to the square of a given right line (D), not greater than the half (CB) of the first mentioned right line.

On CB, take CE, whose square is equal to the excess of the square of CB above that of D (Cor. 2. 47. 1); and, since the square of CB is equal to the two squares of CE and D (Constr.), taking from each the square of CE, there remains the square of D equal to the excess of the square of CB above that of CE, or, which is equal [5. 2], the rectangle AEB.



Cor. 3.—To add to a given right line (AB) such a part, that the rectangle under the whole and part added, may be equal to the square of a given right line (D).

Bisect AB in C, and on CB produced take CE, whose square is equal to the squares of CB and D together (Cor. 1. 47. 1); and, since the square of CE is equal to the two squares of CB and D together (Constr.), taking from each the square of CB, there remains the square of D equal to the excess of the square of CE above that of CB, or which is equal (6. 2), the rectangle AEB.

Cor. 4.—In equiangular triangles (ABC, DEF, having the angles at A and D equal, as also those at B and E, and at C and F), the rectangles under the sides about any of the equal angles (as ABC and E) taken alternately, are equal, (namely, the rectangles, formed of sides, taken from different triangles and opposed to contrary angles, as under AB, EF, and under BC, DE).

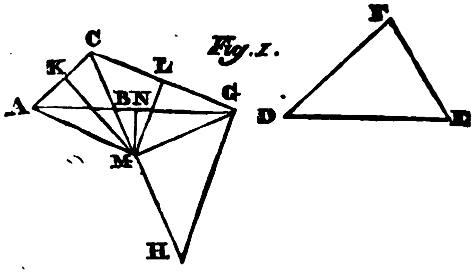


Fig. 1.

In AB produced, take BG equal to EF, and in CB produced, BH equal to DE, and join GH; and, because the angle HBG is equal to ABC (15. 1), or its equal (Hyp.) E, and the sides BG, BH severally equal to EF, ED (Constr.), the triangle HBG is equiangular to the triangle DEF (4. 1), and therefore to the triangle ABC (Hyp. and Ax. 1. 1), having the angles at A and H equal, and at C and G; join CG, bisect AC, CG in K and L, by the perpendiculars KM, LM (10. and 11. 1), meeting each other in M.

Join MA, MG, and draw MN perpendicular to AG (12. 1), and, because ML, LG and the angle MLG, are severally equal to ML, LC and the angle MLC, the right lines MG, MC are equal (4. 1); in like manner MA, MC may be proved equal; and, because the triangle MCG is isosceles, and MA equal to MC, the angle CAG is half of the angle CMG (Cor. 3. 32. 1); and, since CHG has been proved equal to CAG, the angle CHG is also half of CMG, therefore MH is equal to MC (Cor. 4. 32. 1); and, in the isosceles triangle AMG, the perpendicular MN bisects the base AG in N (Cor. 26. 1); and, in the triangle BMG, the rectangle under the sum and difference of MG and MB, or, BH being equal to the sum, and CB to the difference, of these right lines, the rectangle CBH, is equal to the rectangle under the sum and difference of GN, BN, or the rectangle ABG (Cor. 1. 5 and 6. 2); and BH is equal to DE, and BG to to EF, therefore the rectangle under BC and DE, is equal to the rectangle under AB and EF.

Secondly, let the point M be without the right line CH, as in fig. 2. and 3.

Fig. 2.

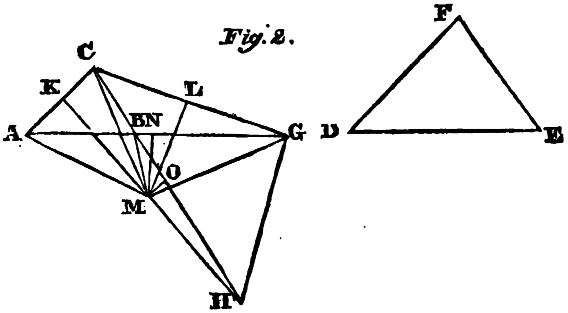
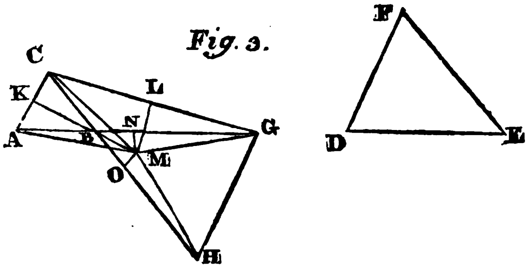


Fig. 3.

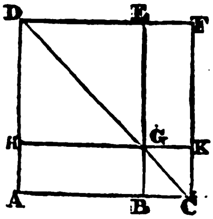


Construct as in the former case, join moreover MB, MC, and draw MO perpendicular to CH; and, as before, the triangles ABC, HBG may be proved equiangular, and the right lines MA, MC, MG equal; and, because the triangle CMG is isosceles, and MA equal to MC, the angle CAG is equal to half of CMG (Cor. 3. 32. 1); but CHG is equal to CAG, and therefore also the half of CMG, therefore MH is equal to MC (Cor. 4. 32. 1), and the triangle MCH isosceles, therefore the perpendicular MO bisects the base CH (Cor. 26. 1); and, in the triangle BMH, the rectangle under the sum and difference of BO, OH, or the rectangle CBH, is equal to the rectangle under the sum and difference of BM, MH (Cor. 1. 5 and 6. 2), or of BM, MG, or, which is equal (by the same Cor.), to the rectangle under the sum and difference of BN, NG, or the rectangle ABG; but BH is equal to DE, and BG to EF, therefore the rectangle under BC and DE, is equal to the rectangle under AB and EF.

In like manner it may be proved, that the rectangle under the sides about the equal angles at C and F, or at A and D, of the triangles ABC, DEF, taken alternately, are equal.

PROP. VII. THEOR.

If a right line (AC), be divided into any two parts (AB, BC); the squares of the whole (AC), and one of its parts (BC), are equal to double the rectangle (ACB) under the whole and that part, with the square of the other part (AB).



On AC describe the square ACFD [46. 1], draw CD, through B, draw BE parallel to AD, meeting CD in G, and through G, HK parallel to AC.

The square AF is equal to the rectangles AK, KE with the square HE; add to each, the square BK, the squares AF, BK of AC, BC, are equal to the rectangles AK, BF with the square HE.

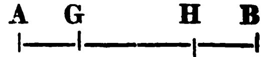
But the rectangles AK, BF are, each of them, equal to the rectangle ACB, because CK is equal to BC, and CF to AC [Cor. 2. 46. 1. and Def. 36. 1], and HE is the square of HG, or its equal [34. 1] AB; therefore the squares of AC, BC, are equal to double the rectangle ACB with the square of AB.

Otherwise.

The square of AC, is equal to the squares of AB, BC with double the rectangle ABC (4. 2); add to each the square of BC, and the squares of AC, BC, are equal to the square of AB with double the square of BC and double the rectangle ABC; but double the square of BC with double the rectangle ABC; is equal to double the rectangle ACB [3. 2]; therefore the squares of AC, BC are equal to double the rectangle ACB with the square of AB.

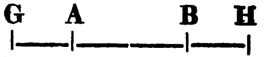
Scholium.—A like observation, as is made in the scholium to the preceding proposition, and 3rd. of this book, is applicable to the 4th proposition and this, that they may be virtually reduced to one, namely; *The sum of the squares of two right lines, is less than the square of their sum, and greater than the square of their difference, by double the rectangle under these right lines;* as is manifest, by supposing, in both propositions, the right lines to be AC, CB.

Corollaries to the preceding propositions of this book.



Cor. 1.—If a right line (AB) be so divided into two points (G, H), that the rectangle (AGB) under the distances of one of these points from its extremes, be equal to the rectangle (AHB) under the distances of the other point from the same extremes, the extreme segments (AG, HB) are equal.

For since the rectangles AGB, AHB are equal (Hyp.), taking from each the rectangle under AG and HB, there remain (by 1. of this,) the rectangles AGH, GHB equal to each other; whence, the side GH being common to both rectangles, the other sides AG, HB are equal (40 and 41. 1).

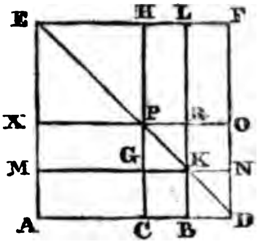


Cor. 2.—If to a right line (AB), there be added on both extremes, such parts (AG, BH), that the rectangles (AGB, BHA) under the added parts (AG, BH,) and the compounds (GB, AH) of the same right line and parts added, be equal; the added parts (AG, BH) are equal.

For since the rectangles AGB, AHB are equal (Hyp.), adding to each the rectangle under AG and BH, the totals, which are (by 1. of this,) the rectangles AGH, BHG, are equal; whence, the side GH being common to both rectangles, the other sides AG, BH are equal (40 and 41. 1).

PROP. VIII. THEOR. (*See note.*)

If a right line (AB), be divided into any two parts (AC, CB); four times the rectangle under the whole (AB), and either part (as CB), with the square of the other part (AC), is equal to the square of the compound of the whole (AB) and part first taken (CB), as of one right line.



On AB produced, take BD equal to CB, and on AD describe the square ADFE [46. 1]; draw DE, through B and C, BL and CH parallel to DF [31. 1], meeting DE in K and P, and through K and P, MN, XO parallel to AB.

Because BN, GR are squares of BD, GK (Cor. 2. 46. 1), or of their equal (Constr. and 34. 1) CB, and CK the square of CB, because BK the side of the square BN is equal to CB, and KO its equal (43. 1), also equal to the square of CB; the four rectangles CK, BN, GR, KO are each equal to the square of CB; again, AG, MP are rectangles under AC CB, because CG, GP are each equal to CB, being sides of the squares just mentioned, and MG to AC (34. 1), and the rectangle PL is equal to MP (43. 1), and therefore equal to the rectangle under AC, CB; also RF is equal to the rectangle under AC, CB, because RO is equal to BD (34. 1.), or CB, and RL is (by the same) equal to PH a side of the square XH, on XP equal to AC; therefore the four rectangles AG, MP, PL, RF are each equal to the rectangle AG under AC, CB; therefore the four squares CK, BN, GR, KO with the four rectangles AG, MP, PL, RF, or, which is equal, the gnomon AOH, is equal to four times the square CK and rectangle AG, or, which is equal (3. 2), four times the rectangle AK; whence, four times the rectangle AK being equal to the gnomon AOH, adding to each the square XH, four times the rectangle AK with the square

XH, is equal to the gnomon AOH and square XH, or, which is equal, to the square AF of AD.

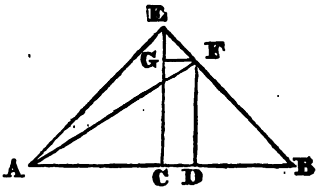
But the rectangle AK is under AB. BC, because BK and BC are equal, being sides of the same square, and XH is the square of AC, because XP is equal to AC (34. 1).

Otherwise, (see the above figure).

On AB produced, take BD equal to BC, and the square of AD is equal to the squares of AB. BD with double the rectangle ABD (4. 2); or, CB and BD being equal, to the squares of AB, BC with double the rectangle ABC; but the squares of AB, BC, are equal to double the rectangle ABC with the square of AC (7. 2), therefore the square of AD, is equal to four times the rectangle ABC with the square of AC.

PROP. IX. THEOR.

If a right line (AB), be divided into two equal parts (AC, CB), and two unequal parts (AD, DB); the squares of the unequal parts (AD, DB), are together double the square of the half line (AC) with the square of the intermediate part (CD).



From C, draw CE perpendicular to AB, and equal to AC or CB (11 and 3. 1); join AE and EB, through D, draw DF parallel to CE, meeting EB in F, and through F, FG parallel to AB, and join AF.

Because the angle ACE, of the triangle ACE, is a right angle, the two angles CAE, CEA together are equal, are equal to a right angle (32. 1); and, because of the equality of AC, CE, are equal to each other (5. 1), and therefore either of them as CEA is half a right angle; in like manner, CEB may be proved to be half a right angle, therefore AEB is a right angle; and the angle EGF is, because of the parallels GF, CD, equal to the internal remote angle ECB (29. 1), and therefore right; whence GEF being half a right angle, the remaining angle EFG, of the triangle EGF, is half a right angle (32. 1), and therefore EG is equal to GF (6. 1); and the angles FDB, DFB are, because of the parallels FD, EC, severally equal to the internal remote angles ECB, CEB (29. 1), therefore FDB is a right angle, and DFB half a right angle, and therefore FBD also half a right angle

(32. 1); whence, the angles DFB, DBF being equal, DF is equal to DB (6. 1): But, because AC, CE are equal, and the angle ACE a right one, the square of AE, which is equal to the two squares of AC, CE (47. 1), is double of one of them, as of the square of AC; also, because EG, GF are equal, and the angle EGF right, the square of EF, which is equal to the two squares of EG, GF (47. 1), is double of one of them, as of the square of GF, or of its equal (34. 1) CD; therefore the squares of AE, EF, or, which, because of the right angle AEF, is equal (47. 1), the square of AF, is double the squares AC, CD; but the square of AF is, because of the right angle ADF, equal to the squares of AD, DF (47. 1), or, DF, DB being equal, to the squares of AD, DB; therefore the squares of AD, DB together, are double the squares of AC, CD together.

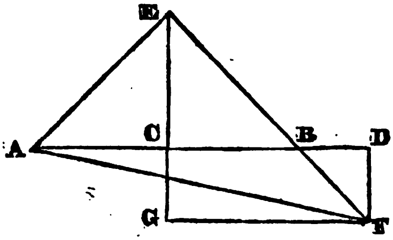
Otherwise, (see the above figure).

The square of AD, is equal to the two squares of AC, CD with twice the rectangle ACD (4. 2), or, AC, CB being equal, to the two squares of AC, CD with twice the rectangle BCD; adding to each the square of DB, the squares of AD, DB, are equal to the squares of AC, CD, DB with twice the rectangle BCD; but twice the rectangle BCD with the square of DB is equal to the squares of CB and CD (7. 2), or, AC, CB being equal, to the squares of AC and CD; therefore the squares of AD, DB, are equal to double the square of AC with double the square of CD.

PROP. X. THEOR.

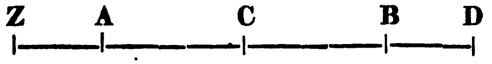
If a right line (AB), be bisected (as in C,) and another right line (BD) be added to it in continuation; the squares of the whole continued line (AD), and of the part added (BD), are together double the squares of the half line (AC) and of the compound (CD) of the half line and part added.

From C, draw CE, perpendicular to AB, and equal to AC or CB (11 and 3. 1); join AE, EB, through D, draw DF parallel to CE, meeting EB produced in F; and through F FG parallel to AB, meeting EG produced in G, and join AF.



Because the angle ACE is right, the two angles CAE, CEA are together equal to a right angle (32. 1), and, because of the equality of AC, CE, are equal to each other (5. 1), and therefore either of them as CEA is half a right angle; in like manner, CEB may be proved to be half a right angle; therefore AEB is a right angle; and because CE, DF are parallel, the angles BDF, BFD are severally equal to their alternates BCE, BEC [29. 1), therefore BDF is a right angle, and BFD half a right angle, and therefore DBF half a right angle (32. 1), and DB equal to DF (6. 1); and because the angle FEG is half a right angle, and the angle G, being equal to its opposite angle D (34. 1), a right angle, EFG is half a right angle, and therefore GE, GF are equal (6. 1): But, because AC, CE are equal, and the angle ACE a right one, the square of AE, which is equal to the two squares of AC, CE (47. 1), is double to either of them, as the square of AC; and, because GE, GF are equal, and the angle G right, the square of EF, which is equal to the two squares of EG, GF, is double to either of them, as to the square of GF, or of its equal (34. 1) CD; therefore the squares of AE, EF, or, which, because of the right angle AEF, is equal (47. 1), the square of AF, is double the squares of AC, CD; but the square of AF is, because of the right angle ADF, equal to the squares of AD, DF (47. 1), or BD, DF being equal, to the squares of AD, DB; therefore the squares of AD, DB together, are double the squares of AC, CD together.

Otherwise.

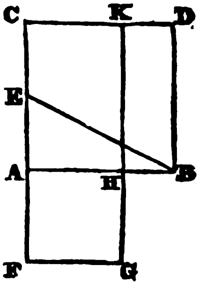


On BA produced beyond A, take AZ equal to BD; the right line ZD is, because of the equals ZA, BD, and AC, CB, bisected in C, and divided unequally in B; therefore the squares of ZB, BD are double the squares of ZC, CB (9. 2); but, because ZA is equal to BD, ZB, AD are equal, and also ZC, CD (Ax. 2. 1); therefore the squares of AD, DB, are double the squares of CD and CB or AC.

Schol.—A like observation, as is made in the scholiums to the 3rd, 6th, and 7th propositions of this book, is applicable to this 10th proposition and the preceding, which may be virtually included in this one, namely: *The squares of the sum and difference of two right lines, are together double the squares of the* *right lines themselves;* as may appear, by supposing, in the figures to both propositions, the right lines to be AC, CD.

PROP. XI. PROB.

To divide a given finite right line (AB) so into two parts, that the rectangle under the whole and one part, may be equal to the square of the other part.



At A draw AC perpendicular and equal to AB (11 and 3. 1), bisect AC in E, join EB, and on EA produced take EF equal to EB; on AB take AH equal to AF; the square of the part AH, is equal to the rectangle under AB and the other part HB.

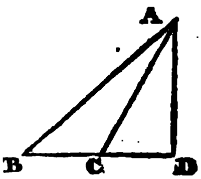
Complete the square AD of AB (46. 1), through H, draw KG, parallel to FC, and through F, FG parallel to AB, meeting KG in G.

Because AC is bisected in E and AF added to it, the rectangle CFA with the square of EA, is equal to the square of EF (6. 2), or of EB, its equal (Constr.), or, to the squares of EA and AB, which are equal to the square of EB (47. 1); taking from each the square of EA, the rectangle CFA remains equal to the square of AB (Ax. 3. 1); but CG is the rectangle under CF, FA, because FG, equal to AH (34. 1), is equal to AF (Constr.); from the rectangle CG, and the square AD of AB, thus proved equal, taking away AK, which is common, the residues AG and HD are equal (Ax. 3. 1); but, because AF is equal to AH, and the angle FAH right, AG is the square of AH, and HD is the rectangle under AB, HB, because BD is equal to AB; therefore AB is so divided in H, that the square of one part AH, is equal to the rectangle under AB and the other part HB, as was required.

PROP. XII. THEOR.

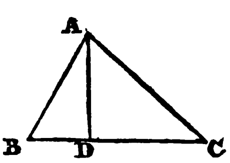
In obtuse angled triangles (ABC, obtuse angled at C), the square of the side (AB) opposite the obtuse angle (ACB), is greater than the squares of the sides (AC, CB) including the obtuse angle, by twice the rectangle under either of these sides (as BC), and the external part (CD) of the same side produced, between a perpendicular (AD) let fall on it from the opposite angle, and the obtuse angle.

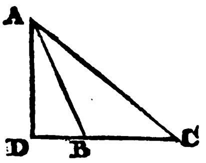
The square of BA is equal to the squares of BD, DA (47. 1), and the square of BD is equal to the squares of BC, CD with twice the rectangle BCD (4. 2); therefore the square of AB is equal to the squares of BC, CD and DA with twice the rectangle BCD; but the square of AC is equal to the squares of CD, DA (47. 1), therefore the square of AB is equal to the squares of BC, CA with twice the rectangle BCD; and so the square of AB is greater than the squares of BC, CA, by twice the rectangle BCD.

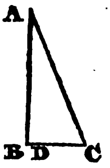


PROP. XIII. THEOR.

In any triangle (ABC, see all the figures to this prop.), the square of a side (AB) subtending an acute angle (C), is less than the squares of the sides (AC, CB) including that angle, by twice the rectangle under either of the sides including that angle (BC), and the right line (DC), intercepted between the perpendicular (AD) let fall on that side, produced, if necessary, from the opposite angle, and the acute angle.





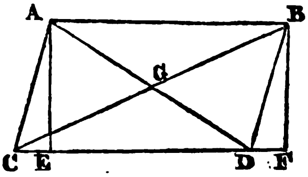


The squares of BC and CD are equal to twice the rectangle BCD with the square of BD (7. 2); adding to each the square of AD, the squares of BC, CD, and AD, are equal to twice the rectangle BCD with the squares of BD and AD; but the squares of CD, AD are equal to the square of AC (47. 1), and the squares of BD, AD to the square of AB (by the same); therefore the squares of BC, AC are equal to twice the rectangle BCD with the square of AB, and so the square of AB, is less than the squares of BC, AC, by twice the rectangle BCD.

Schol. 1.—The demonstration of the case of the 3rd or right hand figure, when the angle ABC is right, may also be thus: The square of AC is equal to the squares of AB, BC (47. 1); adding to each the square of BC, the squares of AC, BC are equal to the square of AB with twice the square of BC; and so the square of AB, is less than the squares of AC, BC, by twice the square of BC.

Schol. 2.—This proposition, the preceding, and the 47th of the 1st book, may be all comprised in one proposition, thus:—*The difference of the square of any side of a triangle, from the squares of the other two sides taken together, is equal to double the rectangle, under either of the sides including the angle opposite the first mentioned side, and the segment of that side, produced, if necessary, between the same angle, and a perpendicular let fall thereon, from the opposite angle; the square of the first mentioned side being equal to, or greater or less than, the squares of the other two sides, according as the angle opposite thereto, is equal to, or greater, or less than, a right angle.*

Schol. 3.—The diagonals (AD, CB) of a parallelogram bisect each other.



Let G be the point in which they meet; because the triangles AGB, DGC have the angles at G equal (15. 1), the angle BAG equal to its alternate CDG (29. 1), and the sides AB, CD equal (34. 1); AG is equal to GD (26. 1). In like manner, CG, GB may be proved equal.

Cor. 1.—The squares of the diagonals (AD, CB) of a parallelogram (AD) taken together, are equal to the squares of all its sides (AB, BD, CD and CA).

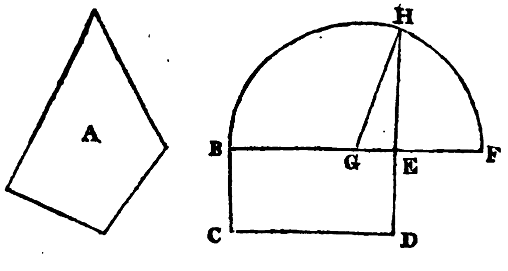
On CD, produced as necessary, let fall the perpendiculars AE, BF (12. 1); and in the triangles ACE, BDF, the angles CEA, DFB are equal, being right angles, and, because of the parallels AC, BD, the external angle BDF is equal to the internal remote ACE (29. 1), and the sides AC, BD opposite the right angles at E and F, are equal (34. 1); therefore CE is equal to DF (26. 1): But the square of AD, opposite the acute angle ACD of the triangle CAD, and twice the rectangle DCE are equal to the squares of AC CD (13. 2); and the square of CB, opposite the obtuse angle CDB of the triangle CBD, is equal to the squares of CD, DB, and twice the rectangle CDF [12. 2]; or, CE, DF being equal, as also AC, BD [34. 1], to the squares of AC, CD and twice the rectangle DCE; whence, adding equals to equals, the squares of AD, CB with twice the rectangle DCE, are equal to double the squares of AC, CD with twice the rectangle DCE; taking from each, twice the rectangle DCE, there remain the squares of AD and CB, equal to double the squares of AC and CD, or, AB, BD being severally equal to CD, CA, to the squares of AB, BD, CD and CA.

Cor. 2.—The squares of the sides [AC, AB] of a triangle [ACB], are double the squares of half the base [CB], and of a right line [AG], drawn to the middle [G] of the base, from the vertical angle [CAB].

Complete the parallelogram ACDB [Cor. 6. 34. 1]; the squares of its sides, being equal to the squares of its diagonals AD, CB [by the preced. cor.], the squares of AC and AB are equal to half the squares of AD, CB; but the squares of AD, CB are fourfold the squares of their halves [Cor. 4. 2], and AG, CG are the halves of AD, CB [Schol. 3. above]; therefore the squares of AC and AB, are double the squares of CG and AG.

PROP. XIV. PROB.

To constitute a square, equal to a given rectilineal figure (A).



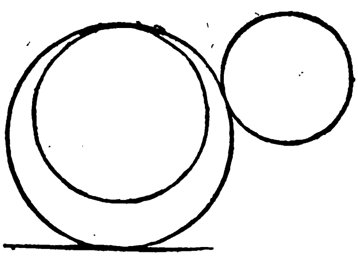
Make the right angled parallelogram BCDE equal to A [45. 1]; and if its adjacent sides BE, ED are equal, it is a square, and what was proposed is done.

If not, on either of these sides as BE, produced, take EF equal to the other ED; bisect BF in G, and from the centre G, at the distance GB or GF, describe the semicircle BHF; let DE be produced to meet the semicircle in H; the square described on EH, is equal to the given rectilineal figure A.

For, drawing GH, because BF is bisected in G, and divided unequally in E, the rectangle BEF with the square of GE, is equal to the square of GF (5. 2); or, GH, GF being equal (Def. 10. 1), to the square of GH; and therefore (47. 1), to the squares of GE and EH; taking from each the common square of GE, the rectangle BEF is equal to the square of EH (Ax. 3. 1); but the rectangle BEF is equal to the rectangle BD, because ED is equal to EF, and so the square described on EH is equal to the rectangle BD, and therefore to the rectilineal figure A.

BOOK III.

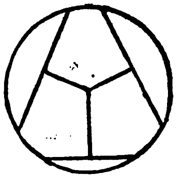
DEFINITIONS.



1. A RIGHT line, is said to touch a circle, or to be a tangent to it, which meeting it, and being produced, does not cut it.

2. Circles, are said to touch one another, which meet, but do not cut each other.

3. A right line, is said to be inscribed in a circle, when its extremes are in the circumference of the circle.

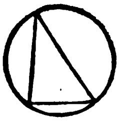


4. Right lines, are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal. *See def.* 21. *Book* 1.

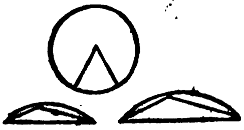
5. And the right line, on which the greater perpendicular falls, is said to be more remote from the centre.

6. A segment of a circle, is a figure contained by a right line, and the part of the circumference it cuts off.

7. The angle of a segment, is that which is included by the right line and the circumference.



8. An angle in a segment, is an angle, contained by two right lines, drawn from any point, in the part of the circumference, by which the segment is bounded, to its extremes.



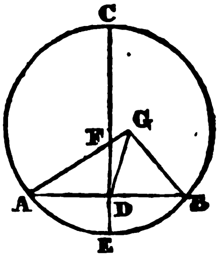
9. An angle, is said to insist, or stand on, the part of the circumference (or arch), included between the legs of the angle.

10. A *sector* of a circle is, a figure, contained by two radiuses, and the part of the circumference between them.

11. Similar segments of circles, are such as receive equal angles.

PROPOSITION I. PROBLEM.

To find the centre of a given circle (ABC).



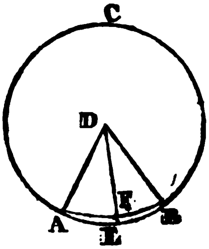
Draw within the given circle any right line AB, which bisect in D (10. 1); from D, draw DC at right angles to AB (11. 1), which produce to meet the circumference in E; bisect EC in F. The point F is the centre of the circle.

No other point in EC, but F, can be the centre, for, if it were, the radiuses drawn from thence to C and E would be unequal, which is absurd (Def. 10. 1): if therefore F be not the centre, let some point, as G, without EC, be, if possible, the centre, and draw GA, GD, GB.

Because, in the triangles GDA, GDB, the side DA is equal to DB (Constr.), DG common, and GA equal to GB (Hyp. and Def. 10. 1), the angles GDA, GDB are equal (8. 1), and therefore right angles (Def. 20. 1); but the angle CDB is a right angle (Constr.), therefore the angles GDB, CDB are equal (Theor. at 11. 1), part and whole, which is absurd (Ax. 9. 1); therefore G is not the centre of the circle. In like manner it may be shewn, that no other point without EC is the centre of the circle; and it is above shewn, that no other point in EC, but F, is the centre; therefore F is the centre.

PROP. II. THEOR.

A right line, which joins any two points (A, B) in the circumference of a circle (ACB), falls wholly within the circle.



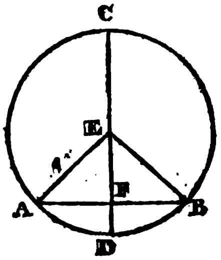
If not, let AEB be a right line, of which a point E falls without the circle; find the centre of the circle D (1. 3), draw DE meeting the circumference in F, and join DA, DB.

Because, in the triangle DAB, the sides DA, DB are equal (Def. 10. 1), the angle DAB is equal to the angle DBA (5. 1); and the external angle DEB, of the triangle AED, is greater than the internal remote angle DAE (16. 1), and therefore than its equal DBE, and so the side DB is greater than DE (19. 1); but DF is equal to DB [Def. 10. 1], therefore DF is greater than DE, the part than the whole, which is absurd: therefore the right line drawn from A to B does not in any part fall without the circle. In like manner it may be proved, that no part of it falls on the circumference, it falls therefore wholly within the circle.

PROP. III. THEOR.

If a right (CD), passing through the centre of a circle (ABC), bisect a right line inscribed in it (AB), not passing through its centre, it cuts it at right angles; and if it cut it at right angles, it bisects it.

Find the centre of the circle E [1. 3], and join EA, EB, which, being radiuses of the circle, are equal [Def. 10. 1], and the triangle EAB is isosceles [Def. 29. 1]; therefore if EF or CD bisect the base AB, it cuts it at right angles, and, if it cut it at right angles, it bisects it [Cor. 26. 1].

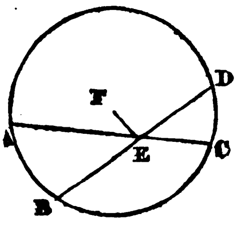


PROP. IV. THEOR.

Two right lines inscribed in a circle, cutting each other, and not passing both through the centre, do not bisect each other.

If one of the right lines pass through the centre, it is manifest, it is not bisected by the other, not passing through the centre.

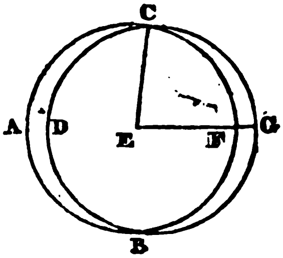
But if neither of them, as AC, BD, pass through the centre, they cannot bisect each other; for let them, if possible, do it, and find the centre of the circle F (1. 3), and join EF; and since AC is bisected in E [Hyp.], FE is perpendicular to AC [3. 3], and the angle FEC right; and since BD is bisected in E, FE is perpendicular to BD [3. 3], and the angle FED right,



and therefore equal to FEC, the part to the whole, which is absurd. Therefore AC and BD do not bisect each other.

PROP. V. THEOR.

If two circles (ACF, DCG) cut each other, they have not the same centre.

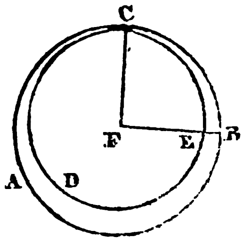


For, if possible, let E be the centre of both circles, and draw EC to the intersection, and EFG meeting the circles in F, G.

Because E is the centre of the circle ACF [Hyp.], EF is equal to EC [Def. 10. 1]; and because E is the centre of the circle DCG, EG is equal to EC [Def. 10. 1]; whence EF, EG, being each equal to EC, are equal to each other [Ax. 1. 1], part and whole, which is absurd; therefore E is not the centre of both the circles ACF, DCG. In like manner it may be shewn, that no other point can be their centre.

PROP. VI. THEOR.

If two circles (ACB, DCE) touch each other internally, they have not the same centre.



For, if possible, let F be the centre of both circles, and draw FC the contact, and FEB meeting them in E and B.

Because F is the centre of the circle DCE [Hyp.], FE is equal to FC [Def. 10. 1]; and because F is the centre of the circle ACB, FB is equal to FC [Def. 10. 1]; whence FE, FB, being each equal to FC, are equal to each other [Ax. 1. 1], part and whole, which is absurd: therefore F is not the centre of both the circles ACB, DCE. In like manner it may be shewn, that no other point can be their centre.

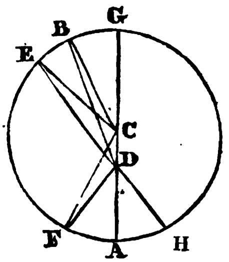
PROP. VII. THEOR.

If any point (D), be taken within a circle (GEA), different from the centre (C); the greatest right line which can be drawn from it to the circumference, is that (DG), which passes through the centre (C).

The remaining part (DA) of the diameter (GA), passing through the point so taken, is the least.

Of others (DB, DE) drawn from that point to the circumference, the right line (DB), which is nearer to that passing through the centre, is greater than one (DE) which is more remote.

And from that point, there can be drawn to the circumference, but two right lines (as DF, DH) equal to each other.



Part 1.—DG passing through the centre, is greater than any other, as DB.

Draw CB, and CG is equal to CB [Def. 10. 1], add to each CD, and DG is equal to DC, CB together; but DC, CB together are greater than DB [20. 1], therefore DG is also greater than DB.

Part 2.—The remaining part DA of the diameter GA is less than any other, as DF.

Draw CF, and CD, DF together are greater than CF [20. 1], and therefore than its equal CA; taking from each CD which is common, DF is greater than DA [Ax. 5].

Part 3.—DB which is nearer to that DG which passes through the centre, is greater than any DE, which is more remote.

Draw CE, and in the triangles BCD, ECD, the sides BC, CD are severally equal to the sides EC, CD, but the angle BCD is greater than the angle ECD, the whole than its part, therefore the base BD is greater than the base ED [24. 1]. In like manner ED may be proved greater than FD.

Part 4.—More than two equal right lines cannot be drawn from that point to the circumference.

For however three right lines be drawn from D to the circumference, either one of them is part of the diameter, and therefore greater or less than either of the others, by part 1st and 2nd; or two of them are on the same part of the diameter, and therefore unequal, by part 3rd.

PROP. VIII. THEOR.

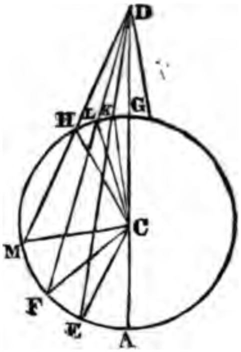
If from any point (D) without a circle, right lines be drawn to the circumference of a circle (GMA); of those drawn to the concave circumference, the greatest is that (DA) which passes through the centre (C).

Of the rest, that which is nearer to that through the centre, is greater than the more remote.

But of those which fall on the convex circumference, the least is that, which, being produced, would pass through the centre.

Of the rest, that which is nearer to the least, is less than the more remote.

Only two equal right lines can be drawn from that point to the circumference.



Part 1.—Of those which fall on the concave circumference, that DA which passes through the centre, is greater than any other, as DE.

Draw CE, and CE is equal to CA (Def. 10. 1), add to each CD, and DA is equal to DC, CE together; but DC, CE together are greater than DE (20. 1), therefore DA is greater than DE.

Part 2.—That DE, which is nearer to that DA through the centre, is greater than the more remote DF.

Draw CF, and in the triangles DCE, DCF, the sides DC, CE are severally equal to DC, CF, and the angle DCE is greater than DCF, therefore the base DE is greater than the base DF (24. 1). In like manner DF may be proved greater than DM.

Part. 3.—Of those which fall on the convex circumference, that DG, which being produced would pass through the centre, is less than any other, as DK.

Draw CG, CK; and DK, KC are greater than DC (20. 1), taking from them, the equals CK, CG, the right line DK is greater than DG (Ax. 5).

Part 4.—That DK, which is nearer to the least DG, is less than the more remote DL.

Draw CL; and DL, LC together, are greater than DK, KC together (21. 1); taking from them the equals CL, CK, the right line DL is greater than DK (Ax. 5. 1). In like manner it may be proved, that DL is less than DH.

Part 5.—Only two equal right lines can be drawn from D to the circumference.

For however three right lines be drawn from D to the circumference, either one of them, produced if necessary, passes through the centre, and is therefore either greater or less than either of the others, by parts 1 and 3; or two of them are on the same part of the diameter, and therefore unequal, by parts 2 and 4.

PROP. IX. THEOR.

If from any point within a circle, more than two equal right lines can be drawn to the circumference, that point is the centre of the circle.

For if it were not the centre, only two equal right lines could be drawn from it to the circumference (7. 3), which is contrary to the hypothesis.

PROP. X. THEOR.

One circle (BAF) cannot cut another (BDF) in more than two points.

Fig. 1.

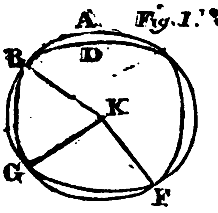
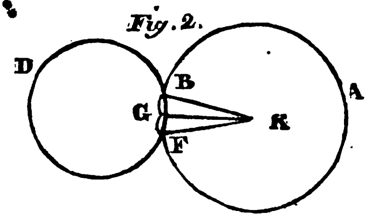


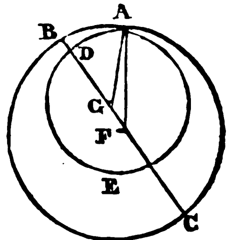
Fig. 2.



If possible, let them cut each other in more than two points, as B, G, F; find the centre K of the circle BAF (1. 3), and draw KB, KG, KF, which are equal (Def. 10. 1); therefore, in the case of figure 1, when K is within the circle BDF, K is the centre of the same circle BDF (9. 3), therefore the circles BAF, BDF, cutting each other, have a common centre, which is absurd (5. 3); therefore the circles cannot cut each other in more than two points in this case. In the case of figure 2, when the centre of each is without the other, more than two equal right lines, as KB, KG, KF are drawn to the circumference of the circle BDF from a point K without it, which is absurd (8. 3); therefore neither in this case can the circles cut each other in more than two points.

PROP. XI. THEOR.

If two circles (ABC, ADE), touch each other internally, the the right line which joins their centres, being produced, passes through their contact.

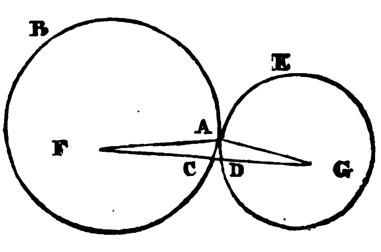


For if not, let the right line BDC joining the centres, cut the circles in D, B, the centre of the circle ABC being F, and that of ADE, G; and draw AF, AG.

The sides AG, GF of the triangle AGF are greater than AF (20. 1), or, than its equal [Def. 10. 1] FB, taking from each GF, which is common, AG is greater than GB; but because G is the centre of the circle ADE, GD is equal to GA; therefore GD is greater than GB, the part than the whole, which is absurd; therefore the right line joining the centres of the circles ABC, ADE cannot fall otherwise than on the contact A, and must therefore pass through it.

PROP. XII. THEOR.

If two circles (ABC, AED) touch each other externally, the right line joining their centres, passes through their contact.



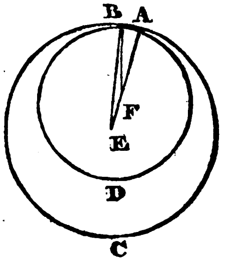
For if not, let F and G be their centres, and let the right line FG joining them, not pass through their contact A, but meet the circles in C and D, and join FA, AG.

Because FA and AG are greater than FG (20. 1), and FC equal to FA, and DG to

AG [Def. 10. 1], therefore FC and DG together are greater than FG, the part than the whole, which is absurd. Therefore the right line, which joins the centres of the circles ABC, AED, cannot pass otherwise than through the contact A, and, of course, passes through it.

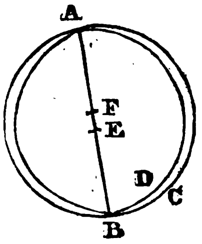
PROP. XIII. THEOR.

One circle cannot touch another, either within or without, in more points than one.

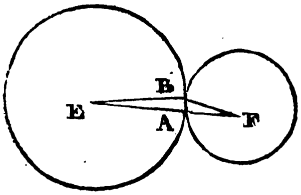


Let the circles AC, AD, if possible, touch each other inwardly in two points A and B; find the centres E, F of these circles (1. 3), which are different [6. 3]; draw EF, which produce to pass through one of the contacts as A (11. 3), and draw EB and FB.

Because FA is equal to FB (Def. 10. 1), adding to each FE, AE is equal to BF, FE; but BF, FE are together greater than BE [20. 1], therefore AE is greater than BE; but, because E is the centre of the circle AC, AE is equal to BE (Def. 10. 1); therefore AE is both equal to, and greater than, BE, which is absurd.



But if the two points of contact A, B be at opposite parts of the right line joining the centres E, F, the right line AB is a diameter of both the circles AC and AD, and AF is equal to FB (Def. 10. 1), and therefore greater than EB, of course AE is greater than EB; but AE is also equal to EB (Def. 10. 1), which is absurd.



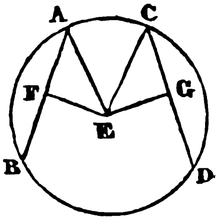
Lastly, let the circles AC, AD, if possible, touch each other externally in two points A, B; draw EF joining the centres E, F, and passing through one of them A [11. 3], and join EB, BF.

Then is EA equal to EB, and AF to BF (Def. 10, 1); therefore EF one side of the triangle EBF is equal to the other two sides EB BF, which is absurd (20. 1).

In no case therefore, do two circles touch each other in more points than one.

PROP. XIV. THEOR.

Equal right lines (AB, CD), inscribed in a circle (ABDC), are equally distant from the centre; and those, which are equally distant from the centre, are equal.



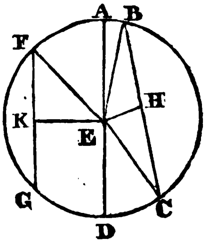
Take E the centre of the circle ABDC (1. 3), join EA, EC, and draw EF, EG perpendiculars to AB, CD.

Because AB, CD are equal (Hyp.), and bisected by the perpendiculars EF, EG (3. 3), the right lines AF, CG are equal (Ax. 7); therefore their squares are equal (Cor. 3. 34. 1); also EA, EC are equal (Def. 10. 1), and therefore their squares [Cor. 3. 34. 1]; but, because the angle AFE is right, the square of AE is equal to the squares of AF, FE [47. 1], and, for the like reason, the square of EC is equal to the squares of CG, GE; therefore the squares of AF, FE are equal to the squares of CG, GE; taking from each the equal squares of AF, CG, the squares of EF and EG are equal [Ax. 3], and therefore the right lines EF, EG themselves [Cor. 1. 46. 1], and, of course, AB, CD are equally distant from the centre [Def. 4. 3].

Let now AB, CD be equally distant from the centre, and, of course, EF, EG equal [Def. 4. 3], their squares are also equal (Cor. 3. 34. 1); and because EA, EC are equal, their squares are equal (Cor. 3. 34. 1); but the square of EA is equal to the squares of EF, FA, and the square of EC to the squares of EG, GC [47. 1]; therefore the squares of EF, FA are equal to the squares of EG, GC; taking away the equal squares of EF, EG, the squares of AF, CG are equal [Ax. 3], and therefore these right lines themselves [Cor. 1. 46. 1]; but, because EF, EG bisect AB, CD [3. 3], the right lines AB, CD are double of the equals AF, CG, and therefore equal to each other [Ax. 6].

PROP. XV. THEOR.

The diameter (AD) is the greatest right line in a circle (AB, CD); and, of all others, that (BC) which is nearer to the centre (E), is greater than one more remote (FG); and the greater (BC) is nearer to the centre than the less (FG).



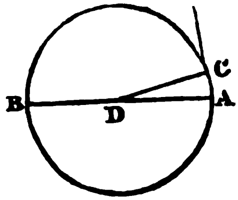
From the centre E draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; the right line EA is equal to EB, and ED to EC (Def. 10. 1), therefore AD is equal to EB and EC; but EB and EC are greater than BC (20. 1), therefore AD is greater than BC.

Let now BC be nearer the centre than FG, and, of course, EH less than EK (Def. 5. 3), the right line BC is greater than FG. For the squares of BH, HE are together equal to the square of EB (47. 1), or of EF, its equal (Def. 10. 1); also the squares of FK, KE are equal to the square of EF [47. 1]; therefore the squares of BH, HE are equal to the squares of FK, KE; whence, the square of EH being less than the square of FK, because EH is less than EK, the square of BH is greater than the square of FK, and, of course, BH greater than FK; whence, BC, FG being doubles of BH, FK [3. 3], the right line BC is greater than FG.

Lastly, let BC be greater than FG, then is BC nearer to the centre than FG. For, because BC is greater than FG [Hyp.], and BH, FK are the halves of BC, FG (3. 3), the right line BH is greater than FK; whence, the squares of BH, HE, and of FK, KE, being each equal to the square of the radius EB or EF [47. 1], and therefore to each other, the square of EH is less than the square of EK, and, of course, EH less than EK, and BC nearer to the centre than FG [Def. 5. 3].

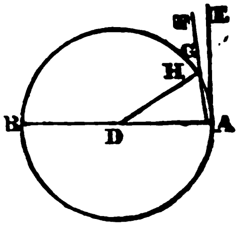
PROP. XVI. THEOR.

A right line, drawn from the extremity (A), of a diameter (AB), of a circle (ABC), perpendicular to it, falls entirely without the circle, and no right line can be drawn, between that right line and the circumference, so as not to cut the circle.



Part 1.—For if the right line, drawn from A at right angles to AB, does not fall entirely without the circle, let it, if possible, meet the circumference in some other point, as C, and from the centre D draw DC.

Because, in the triangle DAC, the sides DA, DC are equal (Def. 10. 1), the angles DAC, DCA are also equal (5. 1), but DAC is a right angle (Hyp.), therefore DCA is a right angle, and the angles DAC, DCA together equal to two right angles, which is absurd (17. 1). Therefore a right line drawn from A, at right angles to AB, does not meet the circle in any other point, and therefore falls entirely without it.



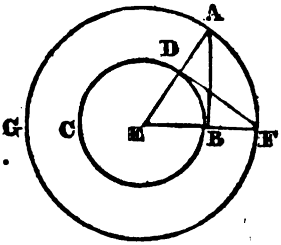
Part 3.—Let now the right line AE be that which is drawn from A, at right angles to AB, which, by part 1. of this, falls entirely without the circle; there cannot be drawn between AE and the circumference, a right line, so as not to cut the circle.

For, if possible, let AF be such, and the angle DAE being right, DAF is acute; from D draw DG perpendicular to AF, meeting the circumference in H; and since, in the triangle DAG, the right angle DGA is greater than the acute angle DAG, the side DA is greater than DG (19. 1); but DH is equal to DA (Def. 10. 1); therefore DH is greater than DG, the part than the whole, which is absurd. Therefore no right line can be drawn between AE and the circle, so as not to cut it.

Cor.—Hence it appears, that a right line, drawn from the extremity of the diameter of a circle, at right angles to it, touches the circle (Def. 1. 3), and that it touches it only in one point.

PROP. XVII. PROB.

From a given point which is not within a given circle (BCD), to draw a right line to touch it.



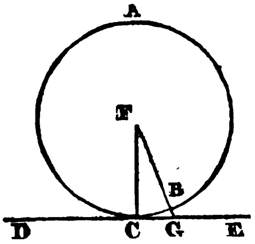
First, let the given point be without the circle, as A; find its centre E (1. 3), join EA, and from the centre E, at the distance EA, describe the circle AFG; through D, draw DF perpendicular to AE (11. 1), meeting the circle AFG in F, and draw EBF and AB; the right line AB touches the circle BCD.

For in the triangles EBA, EDF, the sides EB, EA are severally equal to ED, EF (Def. 10. 1), and the angle E common, therefore the angle EBA is equal to the angle EDF (4. 1); but EDF is a right angle (Constr.), therefore EBA is a right angle, and, of course, AB touches the circle BCD (Cor. 16. 3), being drawn from the given point A.

Let now the given point be in the circumference of the given circle BDC, as the point B; draw BE to the centre E, and BA at right angles to BE; the right line BA touches the circle (Cor. 16. 3).

PROP. XVIII. THEOR.

If a right line (DE) touch a circle (ABC); a right line (FC), drawn from the centre (F), to the contact (C), is perpendicular to the tangent.

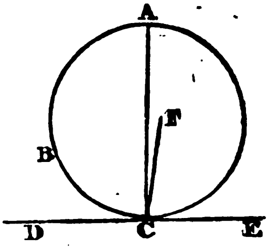


If FC be not perpendicular to DE, from F, draw FBG perpendicular thereto (12. 1), then, because FGC is a right angle, FCG is acute (17. 1), therefore the side FC is greater than FG (19. 1); but FB is equal to FC (Def. 10. 1), therefore FB is greater than FG, the part than the whole, which is absurd; therefore FG is not perpendicular to DE. In like manner it may be shewn, that no other right line but FC, is perpendicular to it, therefore FC is perpendicular to DE.

PROP. XIX. THEOR.

If a right line (DE) touch a circle (ABC); a right line (CA) drawn through from the contact (C), perpendicular to the tangent, passes through the centre.

If not, let the centre be, if possible, without CA, as at F, and join CF.

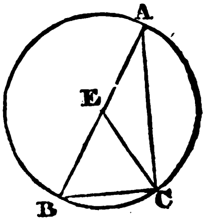


Because FC is drawn from the centre to the contact, it is perpendicular to DE (18. 3), therefore the angle FCE is a right angle; but the angle ACE is a right angle (Hyp.); therefore the angle FCE is equal to ACE, the part to the whole, which is absurd; therefore F is not the centre of the circle. In like manner it may be shewn, that no other point without CA is the centre of the circle, therefore that centre is in CA.

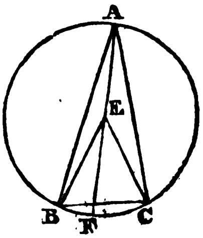
PROP. XX. THEOR.

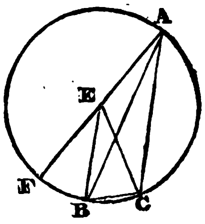
The angle (BEC), at the centre (E), of a circle (ABC), is double of the angle (BAC) at the circumference, on the same base, or same part of the circumference (BC).

Firstly.—Let one of the legs BA of the angle at the circumference, pass through the centre E; and because, in the triangle ECA, the sides EC, EA are equal, the angles EAC, ECA are equal (5. 1); but the external angle BEC is equal to EAC, ECA together (32. 1), and therefore double of BAC.



Secondly.—Let the centre E be within the angle BAC, and join AE, which produce to meet the circumference in F; and because the angle BEF is double the angle EAB (by part 1), and the angle FEC double the angle EAC (by the same), the whole angle BEC is equal to double the angle BAF with double the angle FAC, and therefore to double the whole angle BAC (Ax. 2. 1).





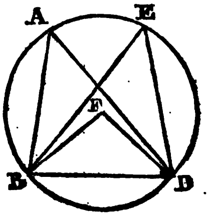
Thirdly.—Let the centre E be without the angle BAC. Join AE, which produce to meet the circumference in F; the angle FEC is equal to double the angle FAC (by part 1), or to double the angle FAB with double the angle BAC; but the angle FEB is equal to double the angle FAB (by the same), which equals being taken away from each of the preceding equals, there remains the angle BEC equal to double the angle BAC (Ax. 3. 1).

Otherwise. (see all the figures to this prop.)

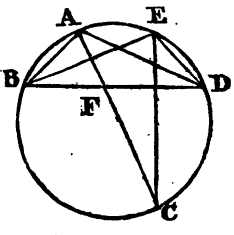
Join BC ; and, because of the equals EB, EC, the triangle EBC is isosceles, and EA is equal to either of the equal sides; therefore the angle BEC is double the angle BAC (Cor. 3. 32. 1).

PROP. XXI. THEOR.

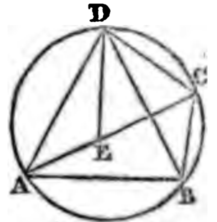
The angles (BAD, BED), in the same segment (BAED) of a circle, are equal.



Find F the centre of the circle BAED (1. 3); and first let the segment BAED be greater than a semicircle, and join BF, FD. Because the angle BFD at the centre, is double of either of the angles BAD, BED at the circumference (20. 3), the angles BAD, BED are equal (Ax. 7).



Secondly, let the segment BAED be, not greater than a semicircle, draw AF to the centre, and produce it to meet the circumference in C, and join CE: and because the segment BADC is greater than a semicircle, the angles in it BAC, BEC are equal (Part I. of this ; and because the segment CBAD is greater than a semicircle, the angles in it CAD, CED are equal (by the same); therefore the angles BAC, CAD together, or the whole angle BAD, and the angles BEC, CED together, or the whole angle BED, are equal (Ax. 2. 1).

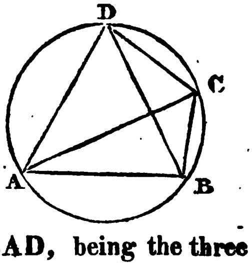


Cor. The rectangles, under the diagonals (AC, BD), of a quadrangle (ABCD) inscribed in a circle, is equal to the rectangles, under the opposite sides, (under AB and CD, and under AD and BC).

Make the angle ADE equal to CDB; to each of which, adding the common angle EDB, the angle ADB is equal to EDC; whence the triangles ADB, EDC, having also the angles ABD, ECD, in the same segment ABCD, equal (by this prop.), are equiangular (32. 1): therefore the rectangle under the sides about the equal angles ABD, ECD, taken alternately, are equal (Cor. 4. 5 & 6. 2), namely, the rectangle under AB and DC, to the rectangle under DB and EC; and the triangles AED, BCD, having the angles ADE, BDC equal [Constr.], and the angles DAE, DBC, in the same segment DABC, equal (by this prop.), are equiangular; therefore the rectangle under the sides about the equal angles DAE, DBC, taken alternately, are equal [Cor. 4. 5 & 6. 2], namely, the rectangle under AD and BC to the rectangle under DB and AE; but the rectangles under DB and EC and under DB and AE, are equal to the rectangle under DB and AC (1. 2); therefore the rectangles under AB and DC, and under AD and BC, are together equal to the rectangle under DB and AC.

PROP. XXII. THEOR.

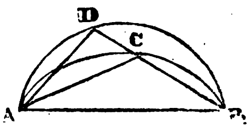
The opposite angles, of a quadrangle (ABCD), inscribed in a circle (ABCD), are equal to two right angles.



Draw AC, DB, and the angles ABD ACD, being in the same segment ABCD, are equal [21. 3]; also, the angles ACB, ADB, being in the same segment ADCB, are equal [21. 3]; therefore the two angles ACB, ACD, or the whole BCD, are equal to the two angles, ABD, ADB; adding to each the angle BAD, the angles BCD, BAD are equal to the angles ABD, ADB and BAD; but the angles ABD, ADB, BAD, being the three angles of the triangle ABD, are equal to two right angles [32. 1]; therefore the angles BCD, BAD, are also equal to two right angles. In like manner, the angles ABC, ADC may be proved, to be equal to two right angles.

PROP. XXIII. THEOR.

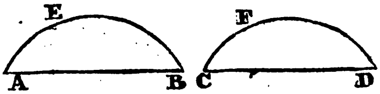
On the same right line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with each other.



For, if possible, let two similar segments of circles, as ACB, ADB be on the same right line AB, and on the same side of it, not coinciding with each other: Then, since the circles ACB, ADB cut each other in the points A, B, they cannot cut each other in any other point [10. 3]; one of the segments therefore must fall within the other; let ACB fall within ADB, and from any point D, of the circumference ADB, draw DB, meeting the circumference ACB in C, and join AD, AC; and because the segments ACB, ADB are similar, (Hyp.), the angle ACB is equal to the angle ADB (Def. 11. 3), an exterior angle of the triangle ACD, to an interior remote, which is absurd [16. 1]. Therefore there cannot be, on the same side of the same right line, two similar segments of circles, which do not coincide.

PROP. XXIV. THEOR.

Similar segments of circles (AEB, CFD), on equal right lines (AB, CD,) are equal.



For if the segment AEB be so applied to the segment CFD, that the point A may be on the point C, and the right line AB on CD, the point B would fall on D, because AB is equal to CD, and the right line AB would coincide with CD; therefore the segment AEB, being similar to the segment CFD (Hyp.), would coincide with it [23. 3]; therefore the segments AEB, CFD are equal (Ax. 8. 1).

Cor. 1. From the proof of this proposition, it follows; that the circumferences AEB, CFD, of similar segments of circles, on equal right lines, are also equal, since they also would coincide.

Cor. 2. And, by a similar argument, as is used in this proposition, it may be proved; that circles, having equal diameters or semi-diameters, are equal; for if they be so applied to each other, that their centres coincide, since their semi-diameters are equal, their circumferences coincide; whence the circles themselves and their circumferences are equal.

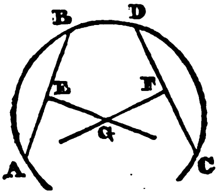
Cor. 3. Circles, having unequal diameters or semidiameters, are unequal, those having the greater diameters or semi-diameters, being the greater.

Cor. 4. And, of unequal circles, the greater has the greater diameter or semi-diameter; for if the diameters were equal, the circles would be equal [Cor. 2. 24. 3], contrary to the hypothesis; and if the diameter of the former were the less, the circle would be less [Cor. 3. 24. 3]; which is also contrary to the hypothesis.

Cor. 5. Equal circles have equal diameters and semidiameters; for if the diameters or semidiameters were unequal, the circles would be unequal (Cor. 3. 24. 3), contrary to the hypothesis.

PROP. XXV. PROB.

A segment of a circle (ABC) being given, to describe the circle of which it is a segment.

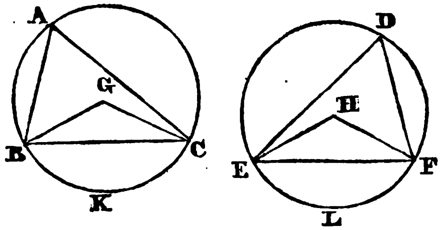


Let two right lines AB, CD, not parallel to each other, and terminated by the circles, be drawn; bisect these right lines in E and F, and, through the points of bisection, draw EG, FG, at right angles to AB, CD, meeting each other in G; their intersection G, is the centre of the circle, whereof ABC is a segment.

Because the right line AB, terminated in a circle, is bisected by a perpendicular EG, the right line EG passes through the centre (Proof of 1. 3); for the same reason, FG passes through the centre; therefore their intersection G, is the centre of the circle, of which ABC is a segment; whence the circle itself may be described.

PROP. XXVI. PROB.

In equal circles (ABC, DEF), equal angles, stand on equal circumferences or arches; whether they be at the centres (as G, H), or at the circumferences (as A, D).



If the angles G, H be at the centre, let there be constituted the angles A, D at the circumferences on the same arches, and join BC, EF.

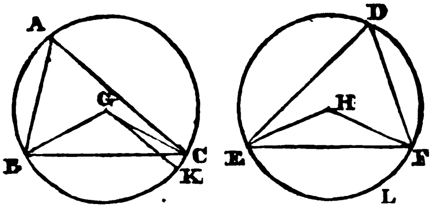
Because the circles ABC, DEF are equal, their semi-diameters are equal (Cor. 5. 24. 3); therefore, in the triangles BGC, EHF, the sides BG, GC are severally equal to EH, HF, and the angles G, H are equal (Hyp.), therefore the bases BC, EF are equal (4. 1); but the angles BAC, EDF are equal (20. 3 and Ax. 7. 1), therefore the segments BAC, EDF are similar (Def. 11. 3); and they are constituted on equal right lines BC, EF, therefore the circumferences BAC, EDF are equal (Cor. 1. 24. 3); but the whole circumference BACK is equal to the whole circumference EDFL (Hyp. & Cor. 5 & 2. 24. 3), therefore the remaining circumference BKC is equal to the remaining circumference ELF.

If the equal angles, as A, D, be at the circumference, and acute; drawing BG, GC, EH and HF, the arches BKC and ELF may, in like manner, be proved equal.

But if the equal angles at the circumference be either right or obtuse, let them be bisected, their halves are equal (Ax. 7. 1), and it may be shewn as above, that the arches, on which they stand, are equal, and therefore the whole arches are equal.

PROP. XXVII. THEOR.

In equal circles (BAC, EDF), the angles which stand on equal arches (BKC, ELF) are equal, whether they be at the centres (as BGC, EHF), or at the circumferences (as BAC, EDF).

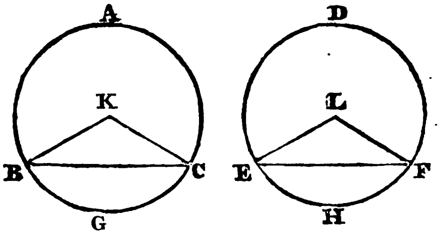


First. The angle BGC is equal to the angle EHF; for if not, let one of them, as BGC, be the greater, and make the angle BGK equal to EHF (23. 1).

Because then, in the equal circles BAC, EDF, the angles BGK, EHF are equal [Constr.], the arches BK, EF are equal [26. 3]; but the arches BC, EF are equal [Hyp.]; therefore the arches BK, BC, being each equal to EF, are equal, part and whole, which is absurd; therefore the angles BGC, EHF are not unequal, they are therefore equal; whence the angles BAC, EDF, being halves of the equal angles BGC, EHF [20. 3], are equal.

PROP. XXVIII. THEOR.

In equal circles (ABC, DEF), equal right lines (BC, EF), cut off equal arches, the greater (BAC), equal to the greater (EDF), and the less (BGC), to the less (EHF).



If the equal right lines be diameters, the proposition is manifest.

But if not, find K and L the centres of the circles [1. 3], and join BK, KC, EL, LF.

Because the circles are equal, the right lines BK, KC are severally equal to EL, LF (Cor. 5. 24. 3), and BC is equal to EF (Hyp.), therefore the angles K, L are equal [8. 1], and therefore the circumference BGC is equal to the circumference EHF [26. 3]; and the whole circumference BACG is equal to the whole circumference EDFH, therefore the remaining circumference **BAC** is equal to the remaining circumference **EDF**.

PROP. XXIX. THEOR.

In equal circles (ABC, DEF, see fig. in the preceding Prop.), equal circumferences (BGC, EHF), are subtended by equal right lines (BC, EF).

If the equal circumferences be semicircles, the proposition is manifest.

If not, find the centres of the circles K, L [[illegible]. 3]; and, because the circumferences BGC, EHF are equal, the angles K, L are equal [27. 3]; and, because the circles ABC, DEF are equal, the right lines BK, KC are severally equal to EL, LF (Cor. 5. 24. 3); whence, the triangles BKC, ELF, having BK, KC and the included angle K, severally equal to EL, LF and the included angle L, the bases BC, EF are equal.

Scholium. What are demonstrated in the four preceding propositions about equal circles, are also manifestly true about the same.

Cor. 1.—In equal circles [ABC, DEF, see fig. to prop. xxvi of this], sectors [BGC, EHF], which stand on equal arches [BKC, ELF], are equal.

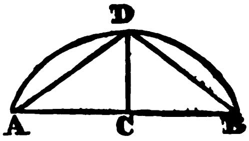
Because the arches BKC, ELF are equal [Hyp.], the right lines BC, EF are equal [by this prop.]; and the angle BAC is equal to EDF [27. 3], and therefore the segments BAC, EDF are similar [Def. 11. 3], and, being on equal right lines BC, EF, are equal [24. 3]; taking each from the whole circles, which are equal [Hyp.], the remaining segments BKC, ELF are equal; and the triangles BGC, EHF, being mutually equilateral, are equal [Cor. 8. 1]; therefore, adding these triangles to the equal segments BKC, ELF the sectors BGC, EHF are equal.

Cor. 2.—In equal circles, equal angles, whether at the centre or circumference, are subtended by equal right lines.

For these equal angles stand on equal arches (26. 3), and the equal arches are subtended by equal right lines (by this prop.).

PROP. XXX. PROB.

To bisect a given circumference or arch of a circle (ADB).

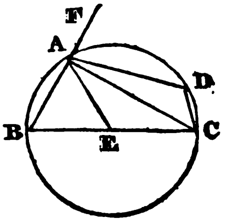


Join AB, which bisect in C (10. 1), from C, draw CD at right angles to AB, which bisects the circumference in D.

Join AD, DB; and in the triangles ACD, BCD, the sides AC, CB are equal (Constr.), CD common to the two triangles, and the angle ACD equal to BCD (Def. 20. 1); therefore the base AD is equal to the base BD (4. 1), and therefore the circumferences AD, DB, which they subtend, are equal [28. 3], and so the given circumference ADB is bisected in D.

PROP. XXXI. THEOR.

The angle (BAC) in a semicircle, is a right angle; but an angle (as ABC), in a segment greater than a semicircle, is less than a right angle; and an angle (as ADC), in a segment less than a semicircle, is greater than a right angle.



Let E be the centre of the circle, join AE, and produce BA, as to F; and because, in the triangle EAB, the sides EA, EB are equal, the angles EAB, EBA are equal (5. 1); and because, in the triangle EAC, EA is equal to EC, the angles EAC, ECA are equal (5. 1); therefore the whole angle BAC is equal to the two angles ABC, ACB [Ax. 2. 1]; but the exterior angle FAC, of the triangle BAC, is equal to the two angles ABC, ACB [32. 1]; therefore the angle BAC is equal to the angle FAC, and therefore a right angle [Def. 20. 1].

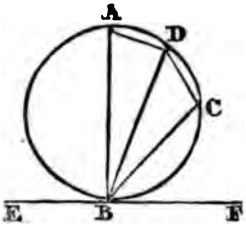
And, because the two angles BAC, ABC, of the triangle ABC, are less than two right angles [17. 1], and BAC is a right angle, the angle ABC, in a segment ABC greater than a semicircle, is less than a right angle.

And, because the two opposite angles ABC, ADC, of the quadrangle ABCD in the circle, are equal to two right angles [22. 3], and ABC is less than a right angle; the angle ADC, in a segment ADC less than a semicircle, is greater than a right angle.

Cor.—A circle, described about the hypothenuse [BC], of a right angled triangle [ABC], passes through the right angle [BAC]; for, if it cut the right line [BF] in any other point but [A], the angle [BAC] would be greater or less, than the angle formed at the intersection, by right lines drawn from it to the extremes of the hypothenuse BC [16. 1], which angle so formed being, by this prop. a right one; the angle [BAC] would be greater or less than a right angle, contrary to the hypothesis.

PROP. XXXII. THEOR.

If a right line (EF) touch a circle (ABCD), and from the contact (B), a right line (BD) be drawn cutting the circle; the angles made by the tangent and cutting line, are equal to the angles in the alternate segments.



If the cutting line pass through the centre, the angles are equal, being right angles [18 and 31. 3].

If not, from the contact B, draw BA at right angles to EF [11. 1], and, having taken any point C, in the circumference BCD, join AD, DC, CB; and because BA is perpendicular to the tangent EF, the centre of the circle is in BA [19. 3], and the angle BDA in a semicircle is a right angle [31. 3], and, therefore, in the triangle ADB, the other two angles BAD, ABD, are equal to a right angle [32. 1], and therefore to the right angle ABF; taking from each the common angle ABD, the angle DBF is equal to the angle BAD in the alternate segment. And, because, in the quadrilateral figure ABCD inscribed in a circle, the opposite angles BAD, BCD are equal to two right angles [22. 3], and therefore to the angles DBF, DBE, which are also equal to two right angles [13. 1]; taking away the equal angles BAD, DBF, the remaining angle DBE is equal to the remaining angle DCB in the alternate segment.

PROP. XXXIII. PROB.

On a given right line (AB), to describe a segment of a circle, which may receive an angle, equal to a given rectilineal angle (C).

First.—Let the given angle C be a right one, [see fig. 1]. Bisect the given right line AB in F [10. 1], from the centre F, at the distance FA, describe the semicircle AHB; the angle AHB in a semicircle is equal to the right angle C [31. 3].

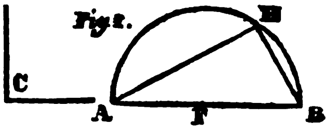


Fig. 1.

Secondly.—Let the angle C not be a right one (see fig. 2 and 3), and at the point A, with the right line AB, make the angle BAD equal to C [23. 1], and from the point A, draw AE at right angles to AD (11. 1); bisect AB in F, from F draw FG perpendicular to AB (11. 1), and join GB; and because, in the triangles AFG, BFG, AF is equal to FB, FG common, and the angles AFG, BFG equal [Theor. at 11. 1], the bases AG, BG are equal [4. 1], and the circle described from the centre G, at the distance GA, passes through B; let this circle be AHB; and, because AD is drawn from the extremity of the diameter AE, perpendicular to it, AD touches the circle [Cor. 1. 16. 3], and therefore the angle BAD, or which is equal [Constr.], the angle C, is equal to the angle in the alternate segment AHB [32. 3]; and so a segment AHB is described on the given right line AB, which may receive an angle, equal to the given angle C, as was required to be done.

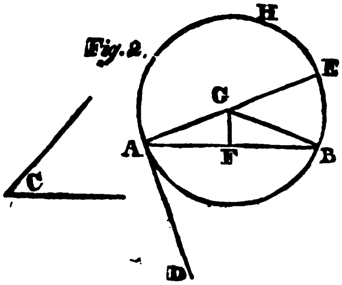


Fig. 2.

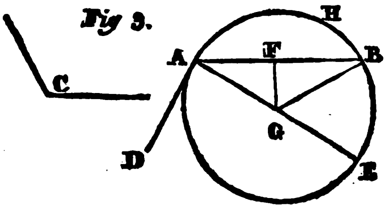
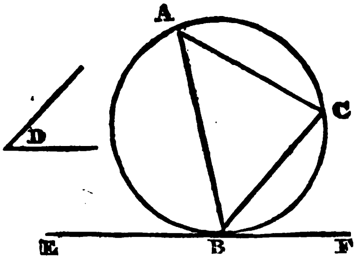


Fig 3.

PROP. XXXIV. PROB.

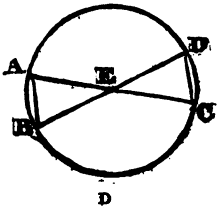
From a given circle (ABC), to cut off a segment, which may receive an angle, equal to a given rectilineal angle (D).



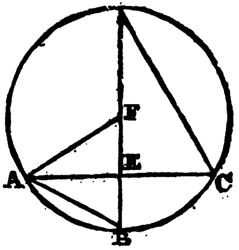
Draw EF touching the circle in any point B (17. 3), and at the point B with the right line BF, make the angle FBC equal to D [23. 1]: and, because EF is a tangent to the circle, the angle FBC, or, which is equal (Constr.), the angle D, is equal to the angle in the alternate segment BAC (32. 3); and therefore, there is cut off from the given circle, a segment BAC, receiving an angle equal to the given angle D, as was required to be done.

PROP. XXXV. THEOR.

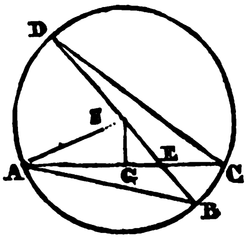
If two right lines (AC, BD), inscribed in a circle (ABCD), cut each other, the rectangle (AEC), under the segments of one, is equal to the rectangle (BED), under the segments of the other.



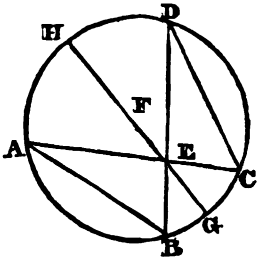
Case 1.—If both pass through the centre; AE, EC, BE, ED being all equal (Def. 10. 1), the rectangle AEC is equal to the rectangle BED (Cor. 3. 34. 1).



Case 2.—If one of them BD, passing through the centre F, cut the other AC, not passing through the centre, perpendicularly, join AF; and because BD is cut equally in F, and unequally in E, the rectangle DEB with the square of EF, is equal to the square of FB (5. 2), or of FA, and therefore, to the squares of AE and EF (47. 1); taking away the common square of EF, the rectangle DEB is equal to the square of AE, or AE, EC being equal (3. 3), to the rectangle AEC.



Case 3.—If one of them DB, passing through the centre F, cut the other AC, not passing through the centre, obliquely, join AF, and draw FG perpendicular to AC; and since DB is divided equally in F, and unequally in E, the rectangle DEB and the square of FE, or, (the squares of FG, GE, being equal to the square of FE 47. 1,), the rectangle DEB, and the squares of FG, GE, are equal to the square of FB (5. 2), or FA, or, which is equal (47. 1), to the squares AG, GF; taking away the common square of FG, the rectangle DEB with the square of GE, is equal to the square of AG; but, because FG is perpendicular to AC, AC is bisected in G (3. 3), therefore the rectangle AEC with the square of GE, is equal to the square of AG [5. 2]; wherefore the rectangle DEB with the square of GE, is equal to the rectangle AEC with the square of GE; taking away the common square of GE, the rectangle DEB is equal to the rectangle AEC.



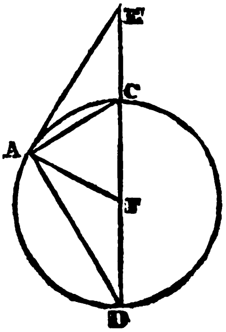
Case 4.—But if neither of the right lines AC, BD pass through the centre, find the centre F, and through E, draw the diameter HG; then the rectangles AEC, DEB, being each of them equal to the rectangle HEG (case 3 of this), are equal to each other.

Otherwise. (*see all the figures to this Prop.*)

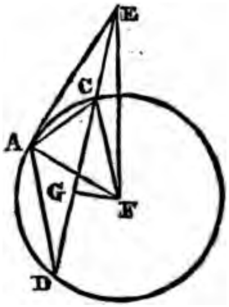
Join AB, CD; and, since the triangles AEB, DEC, having their angles at E equal (15. 1), and the angles ABE, DCE, in the same segment ABCD, also equal (21. 3), are equiangular (32. 1), the rectangles under the sides about the equal angles AEB, DEC, taken alternately, are equal (Cor. 4. 5 & 6. 2), namely, the rectangle AEC to the rectangle DEB.

PROP. XXXVI. THEOR.

If from any point (E) without a circle (CAD), two right lines (ED, EA) be drawn to it, one of which (ED) cuts it, and the other (EA) touches it; the rectangle under the whole cutting line (ED) and the external segment (EC), is equal to the square of the tangent (EA).



Case 1. If ED pass through the centre F, join AF; and, because CD is bisected in F, and CE a part added to it, the rectangle DEC with the square of FC, is equal to the square of FE (6. 2), or which is equal (47. 1), to the squares of FA and AE; taking away the equal squares of FC and FA, the rectangle DEC, is equal to the square of AE (Ax. 3).



Case 2. If ED do not pass through the centre F, draw FG perpendicular to it, and join FA, FC and FE; and, because DC is bisected in G (3. 3), and CE added to it, the rectangle DEC with the square of GC, is equal to the square of GE (6. 2); adding to each the square of GF, the rectangle DEC, and the squares of CG, GF, or, (the squares of CG, GF being equal to the square of FC 47. 1,), the rectangle DEC, and the square of FC, are equal to the squares of EG, GF, or, which is equal (47. 1), to the square of FE, or, which is equal (47. 1), to the squares of FA, AE; taking away the equal squares of FC, FA, the rectangle DEC is equal to the square of AE (Ax. 3).

Otherwise. (*See both figures to this prop.*)

Join DA, AC; and, since the triangles ECA, EAD, having the angle AED common, and the angles EAC, EDA equal (32. 3), are equiangular (32. 1), the rectangles under the sides about the common angle AED, taken alternately, are equal (Cor. 4. 5 and 6. 2), namely, the rectangle CED to the square of EA.

Cor. 1.—Hence, if from any point without a circle, two right lines be drawn cutting it, the rectangles under these right lines and their external segments are equal, being each equal to the square of a tangent, drawn from the same point to the circle.

Cor. 2.—Two tangents drawn to a circle, from any point without it, are equal.

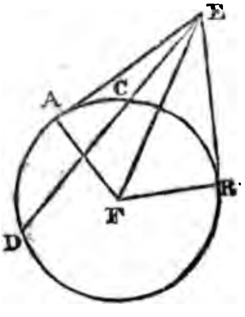
For their squares are equal, being each equal to the same rectangle.

Cor. 3.—From this, and the preceding proposition, it is manifest, that, if a right line, passing through any point, either within or without a circle, cut it in two points or touch it; the square of the segment of the tangent, or rectangle under the segments of the secant, between that point, and the point or points, in which it touches or cuts the circle, is equal to the difference of the squares of the radius, and the distance of the same point from the centre of the circle.

Cor. 4.—Hence also if two right lines, meeting each other, both touch, or both cut, or one of them touch, and the other cut a circle; the squares of the segments of the tangents, or rectangles under the segments of the secants or cutting lines, between their concourse, and the points in which they meet the circle, are equal.

PROP. XXXVII. THEOR.

If from any point (E) without a circle, two right lines (DE, AE) be drawn, one of them (DE) cutting the circle, and the other (AE) meeting it, and the rectangle (DEC) under the cutting line, and its external segment, be equal to the square of the line which meets it; the right line (AE) which meets the circle, touches it.



From E, draw EB touching the circle (17. 3), find the centre F (1. 3), and join FA, FE, FB.

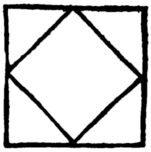
Because EB touches the circle, and DE cuts it, the square of EB is equal to the rectangle DEC (36. 3); but the square of AE is equal to the rectangle DEC (Hyp.), therefore the squares of AE and EB are equal (Ax. 1. 1), and, of course, the right lines AE, EB themselves (Cor. 1. 46. 1); therefore, in the triangles FAE, FBE, the sides FA, AE are severally equal to FB, BE, and FE is common to both triangles, therefore the angles FAE, FBE are equal (8. 1); but FBE is a right angle (18. 3), therefore FAE is also a right angle, and, of course AE touches the circle (Cor. 16. 3).

BOOK IV.

DEFINITIONS.

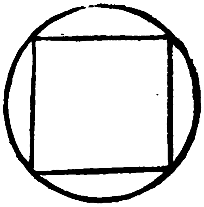
1. A RECTILINEAL figure, is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure, are in the perimeter of the other.

2. A rectilineal figure, is said to be circumscribed about another rectilineal figure, when the perimeter of the former touches all the angles of the other.



3. A rectilineal figure, is said to be inscribed in a circle, when all its angles, are in the circumference of the circle.

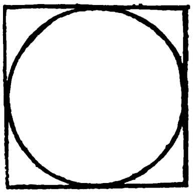
4. A circle, is said to be circumscribed about a rectilineal figure, when every angle of the rectilineal figure, is in the circumference of the circle.



5. A rectilineal figure, is said to be circumscribed about a circle, when every side of it, touches the circle.

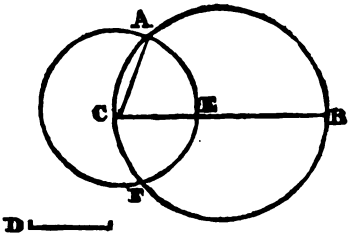
6. A circle, is said to be inscribed in a rectilineal figure, when every side of the rectilineal figure, touches the circle.

7. A regular figure, is that, which is equilateral and equiangular.



PROPOSITION I. PROBLEM.

In a given circle (CAB), from a given point (C) in its circumference, to inscribe a right line, equal to a given right line (D), not greater than the diameter of the circle.



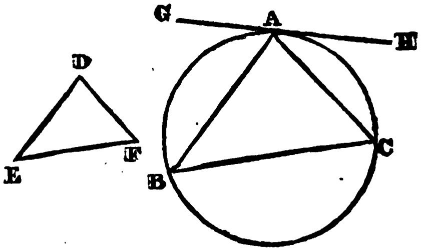
Draw the diameter of the circle CB, and, if this be equal to D, what was required is done.

If not, take from CB, a part CE equal to D (3. 1), and from the centre C, at the distance CE, describe the circle AEF, and to either of its intersections with the given circle, as A, draw CA, this is equal to CE (Def. 10. 1), and therefore to the given right line D (Constr. and Ax. 1. 1).

Cor.—Hence it appears, how, in a given circle, from a given point in its circumference, an arch may be taken, equal to a given arch, of an equal circle; namely, by drawing the chord of the given arch, and inscribing in the given circle, from the given point, a right line equal to that chord (by this prop.), which right line cuts off an arch equal to the given one (by 28. 3).

PROP. II. PROB.

In a given circle (BAC), to inscribe a triangle, equiangular to a given triangle (EDF).

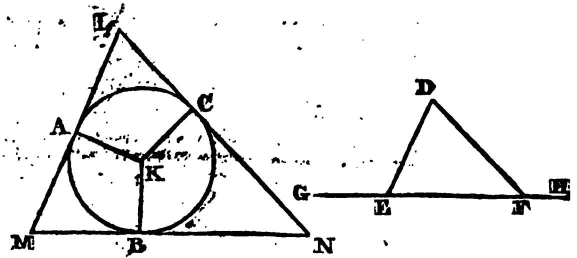


Draw the right line GH, touching the circle in any point A (17. 3), and at the point A, with the right line AH, make the angle HAC equal to the angle E (23. 1); and at the same point, with the right line AG, make the angle GAB equal to angle F, and join BC.

The angle E is equal to the angle HAC (Constr.), or, (32. 3), to the angle B in the alternate segment; for a like reason, the angles F and C are equal; therefore the remaining angle D is equal to the remaining BAC (32. 1); therefore the triangle BAC, which is inscribed in the given circle (Def. 3. 4) is equiangular to the given triangle EDF.

PROP. III. PROB.

About a given circle (ABC), to circumscribe a triangle, equiangular to a given triangle (EDF),

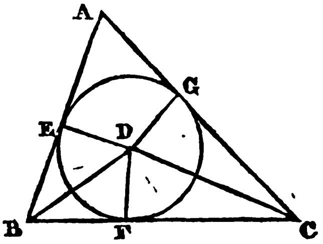


Produce any side EF, of the given triangle, both ways, as to G and H; find the centre K of the given circle (1. 3), from which draw any radius KA, with which, at K, make the angle BKA equal to DEG (23. 1); and, with BK at K, the angle BKC equal to DFH; and draw ML, MN and LN, touching the circle in the points A, B and C (17. 3).

Because the four angles, of the quadrangle MAKB, are equal to four right angles (Cor. 1. 32. 1), and the angles KAM, KBM are right angles (18. 3), the remaining angles AKB, AMB are equal to two right angles, and therefore to DEF, DEG, which together are also equal to two right angles [13. 1]; taking away the equal angles AKB, DEG, the remaining angles AMB, DEF are equal: in like manner, LNM and DFE may be proved, equal; therefore the remaining angle L is equal to the remaining D [32. 1], and, of course, the triangle LMN, which is circumscribed about the given circle [Def. 5. 4], is equiangular to the given triangle DEF.

PROP. IV. PROB.

In a given triangle (ABC), to inscribe a circle.



Bisect any two angles ABC, BCA of the given triangle by the right lines BD, CD (9. 1), which, because the angles DBC, BCD, are together less than ABC, BCA together [Ax. 9], and therefore than two right angles [17. 1], may be so produced, as to meet (Theor. at 29. 1); let them be so produced, and meet, as in D, from which, let fall the perpendicular DE on AB [12. 1], from the centre D, at the distance DE, describe a circle (Post. 3), which is inscribed in the given triangle.

For, the perpendiculars DF and DG being let fall on BC and CA; because, in the triangles DEB, DFB, the angles DEB, DBE are severally equal to DFB, DBF [Constr.], and the side DB common, DE and DF are equal [26. 1]; in like manner, DF, DG may be proved equal; therefore the three right lines DE, DF, DG are equal [Ax. 1. 1], and the circle, described from the centre D, at the distance DE, passes through F and G; and since the angles formed by AB, BC, CA with the radiuses, at the points E, F, G are right angles, these right lines touch the circle (Cor. 16. 3), which is therefore inscribed in the triangle ABC [Def. 6. 4), as was required.

PROP. V. PROB.

About a given triangle (BAC), to circumscribe a circle.

fig. 1.

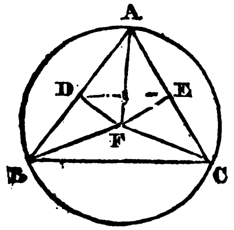


fig. 2.

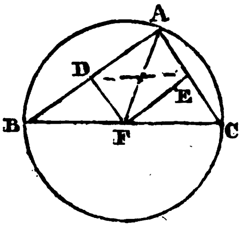
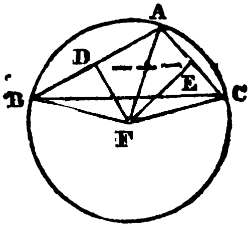


fig. 3.



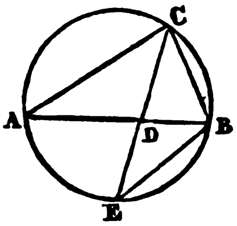
Bisect any two sides AB, AC of the given triangle, in D and E, and from D and E, draw the perpendiculars DF, EF, and join DE. Because the angles ADF, AEF are right angles [Constr.], the angles EDF, DEF are less than two right angles [Ax. 9], therefore the perpendiculars drawn from D and E may be so produced as to meet [Theor. at 29. 1], let them meet as in F, from whence draw, to any angle A of the triangle ABC, the right line FA; from the centre F, at the distance FA, describe a circle, which is circumscribed about the given triangle.

For, FB, FC being drawn, because, in the triangles FDA, FDB, the sides DA, DB are equal (Constr.), FD common, and the angles at D right [Constr.], FA and FB are equal [4. 1]; in like manner, FB and FC may be proved equal, therefore the three right lines FA, FB, FC are equal [Ax. 1. 1], and, of course, a circle, described from the centre F, at the distance FA, passes through B and C, and is therefore circumscribed about the given triangle BAC [Def. 4. 4].

Scholium.—This problem is, in effect, the same, as to describe a circle through three given points, which are not in the same right line.

Cor. 1.—*If the centre (F) of a circle, circumscribing a triangle, be within the triangle (as in fig. 1), all the angles of the triangle are acute; if on any side (BC), of the triangle (as in fig. 2), the angle opposite that side is right; if without the triangle (as in fig. 3), the angle, opposite the side (BC), which is adjacent to the centre, is obtuse.*

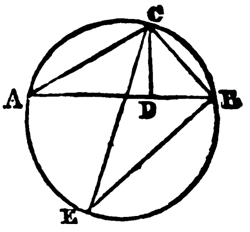
For, in the case of fig. 1, every angle of the triangle ABC, is in a segment greater than a semicircle, and therefore acute [31. 3]; in the case of fig. 2, the angle BAC opposite BC, is in a semicircle, and therefore right [31. 3]; and, in the case of fig. 3, the angle BAC, opposite BC, is in a segment less than a semicircle, and therefore obtuse [31. 3].



Cor. 2.—The rectangle under any two sides [AC, CB] of a triangle [ABC], is equal to the square of a right line [CD], bisecting the angle [ACB] included by them, drawn from that angle, to the opposite side [AB], together with the rectangle [ADB], under the segments of the side [AB], to which the right line is so drawn.

About the triangle ABC circumscribe a circle [5. 4], whose circumference let CD produced meet in E, and join EB.

The triangles ADC, EBC, having the angles ACD, ECB equal [Hyp.], as also the angles CAD, CEB, being in the same segment CAEB [21. 3], are equiangular (32. 1); therefore, the rectangles under the sides about the equal angles ACD, ECB, taken alternately, are equal Cor. 4. 5 and 6. 2), namely, the rectangle under AC and CB, to the rectangle under CD and CE; but the rectangle under CD and CE, is equal to the square of CD with the rectangle CDE (3. 2); and the rectangle CDE is equal to the rectangle ADB (35. 3); therefore the rectangle under AC and CB, is equal to the square of CD, with the rectangle ADB.

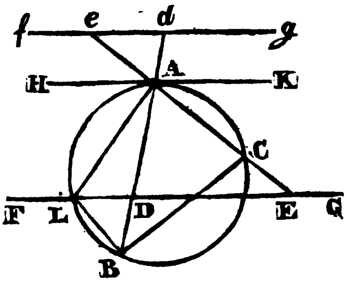


Cor. 2.—The rectangle under any two sides (AC, CB , of a triangle (ABC), is equal to the rectangle under the perpendicular (CD), let fall from the angle (ACB), included by them, on the opposite side (AB), and the diameter of the circle, circumscribed about the triangle.

About the triangle ABC circumscribe a circle (5. 4), and, having drawn its diameter CE, join EB.

The triangles ACD, ECB, having the right angle ADC equal to the angle EBC in a semicircle 31. 3), and the angles CAD, CEB, in the same segment CAEB, equal (21. 3), are equiangular (32. 1); therefore the rectangles under the sides about the equal angles ACD, ECB, taken alternately, are equal (Cor. 4. 5 and 6. 2), namely, the rectangle under AC and CB, to the rectangle under CD and CE.

Theorem.—If two right lines (AB, AC), cutting a circle, and meeting each other in its circumference, meet a right line (FG) parallel to a tangent (HK), drawn through their concourse (A); the rectangles (DAB, EAC), under their segments, between their concourse (A), and the points, in which they meet the circle again, and the parallel, are equal.



Join BC. The angle CBA is equal to the angle EAK (32. 3), or its equal (29. 1) AED; therefore, the triangles ABC, AED, having besides, the angle BAE common, are equiangular; and therefore the rectangles under the sides about the common angle BAE, taken alternately, are equal (Cor. 4. 5 and 6. 2), namely, the rectangle DAB to the rectangle EAC.

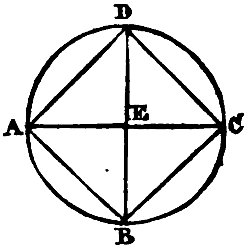
The demonstration is the same, if the right line parallel to the tangent HK, be without the circle, as *fg*; only substituting the small letters *d*, *e*, for their respective capitals, and, instead of the words, "the common angle BAE," using the words, "the equal vertical angles BAC, *eAd*."

Scholium.—If a right line be drawn from A, to a point L, in which FG meets the circle; it may, in like manner, be shewn, that the square of AL is equal to either of the rectangles DAB or EAC; for, drawing LB, the triangles ALD, ABL, having the angle LAB common, and the angle DLA equal to LAH (29. 1), or its equal (32. 3) LBA, are equiangular; and so the rectangles under the sides about the common angle LAB, taken alternately, are equal [Cor. 4. 5 and 6. 2], namely, the rectangle DAB to the square of LA.

PROP. VI. PROB.

In a given circle (ABCD), to inscribe a square.

Draw two diameters AC, BD of the given circle, at right angles to each other, and join AB, BC, CD, DA; ABCD is a square inscribed in the given circle.

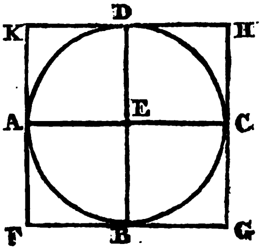


For, since the triangles ABE, BCE, CDE, EDA, have their angles at the centre E equal, being right ones (Theor. at 11. 1), and also the sides containing them EA, EB, EC, ED (Def. 10. 1), the bases AB, BC, CD, DA are equal [4. 1]; therefore ABCD is equilateral, and the angles DAB, ABC, BCD, CDA are right, being angles in a semicircle (31. 3), therefore ABCD is a square (Def. 36. 1), and inscribed in the given circle (Def. 3. 4).

Scholium.—In like manner, as in this proposition, the equality of the angles of the quadrangle ABCD, follows from the equality of the sides, it may be shewn, that any equilateral figure, inscribed in a circle, is also equiangular, and therefore [Def. 7. 4] regular; for each of the angles of the figure stand on an arch composed of the arches subtending all the sides of the figure, except two; which sides being equal [Hyp.], the arches subtending each of them are equal [28. 3], and therefore the whole arches on which the angles stand, and, of course, the angles themselves (27. 3).

PROP. VII. PROB.

About a given circle (ABCD), to circumscribe a square.

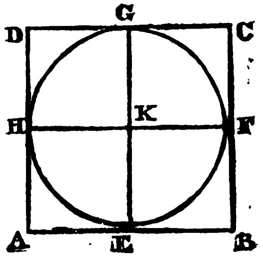


Draw two diameters AC, BD of the given circle, at right angles to each other, and through their extremes A, B, C, D, let tangents to the circle KF, FG, GH, HK be described [17. 3]; FGHK is a square, circumscribed about the given circle.

For since EA is drawn from the centre to the contact, the angle EAF is right [18. 3], but the angle AEB is right [Constr.], therefore FK and BD are parallel [28. 1]; in like manner, GH may be proved parallel to BD, and FG and KH to AC, of course FD, BH, AH, FC are parallelograms [Def. 35. 1]; and, because the angles at A are right, the angles G and H opposite to them are right [34. 1]; in like manner, it may be proved, that the angles K and F are right, therefore the quadrangle FGHK is right angled; and since AC and BD are equal, and FG, KH are each equal to AC, and FK, GH each equal to BD [34. 1], the four sides FG, GH, HK, KF are equal to each other, and the quadrangle FGHK equilateral; it is therefore a square [Def. 36. 1], and circumscribed about the given circle [Def. 5. 4].

PROP. VIII. PROB.

In a given square (ABCD), to inscribe a circle.

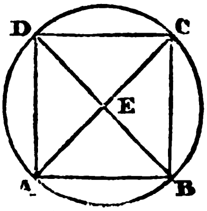


Bisect two adjacent sides AB, AD in E and H; through E, draw EG parallel to AD or BC; and through H, HF parallel to DC or AB, meeting EG in K; a circle described from the centre K, at the distance KE, is inscribed in the given square.

For, since AK, KC, DK and KB are parallelograms [Constr.], the four right lines KE, KF, KG, KH are severally equal to AH, EB, HD, AE [34. 1], and therefore, the latter being halves of the equal sides AB, AD of the given square to each other (Constr.); therefore a circle, described from the centre K, at the distance KH, passes through the points E, F and G; and because the angles at E, F, G and H are right, the sides of the given square, touch that circle in these points (Cor. 16. 3), which is therefore inscribed in the same square (Def. 6. 4).

PROP. IX. PROB.

About a given square (ABCD), to circumscribe a circle.

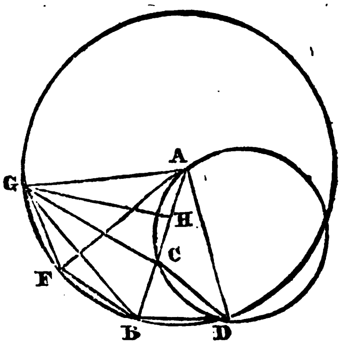


Draw the diagonals AC, BD cutting each other in E; a circle, described from the centre E; at the distance EA, is circumscribed about the given square.

For, because in the isosceles triangle ABD, the angle DAB is right (Def. 36. 1), the angles ABD, ADB are together equal to a right angle (32. 1), and being equal (5. 1), each of them is half a right angle; in like manner, it may be proved, that all the angles, into which the angles of the given square are divided by AC and BD, are halves of a right angle, they are therefore equal to each other (Theor. at 11. 1 and Ax. 7. 1); therefore, in the triangle AEB, because the angles EAB, EBA are equal, the sides EA, EB are equal [6. 1]. In like manner, it may be proved, that EC is equal to EB, and ED to EC, therefore the four EA, EB, EC and ED are equal, and of course, a circle described from the centre E, at the distance EA, passes through B, C and D, and is therefore circumscribed about the given square (Def. 4. 4).

PROP. X. PROB.

To constitute an isosceles triangle, having each of the angles at the base double the vertical angle.



Take any right line AB, and divide it in the point C, so that the rectangle ABC may be equal to the square of AC [11. 2], and, from the centre A, at the distance AB, describe a circle BDG, in which inscribe BD equal to AC [1. 4], and join AD; the isosceles triangle ABD is such as is required, having each of the angles at the base ABD, ADB, double the vertical angle BAD.

Draw DC, and let the circle DCA be circumscribed about the triangle ACD [5. 4].

Because the rectangle ABC is equal to the square of AC [Constr.], or its equal [Constr.] BD, the right line BD touches the circle ACD [37. 3], and therefore the angle BDC is equal to the angle DAC in the alternate segment [32. 3]; adding to each the angle CDA, the angle BDA is equal to CDA, CAD together; but, because the sides AB, AD are equal, the angles ABD, ADB are equal (5. 1), therefore the angle ABD is equal to the angles CDA, CAD together, or, which is equal (32. 1), the angle BCD, and therefore CD is equal BD (6. 1), or its equal CA, and, of course, the angles CAD, CDA are equal (5. 1); whence the angle BDA, or its equal ABD, being equal to CDA, CAD together, is double to CAD, and so an isosceles triangle BAD is constituted, having each of the angles at the base BD, double the vertical angle BAD.

***Cor.* 1.—The greater segment AC, of the radius AB, of a circle, so divided into two parts, that the square of the greater part, is equal to the rectangle under the whole and the other part, is equal to the side of a regular decagon, inscribed in a circle.**

For, since the angle BAD is half of either of the angles ABD, ADB, it is one fourth of both together, and one fifth of all the angles of the triangle ABD, or (32. 1), of two right angles,

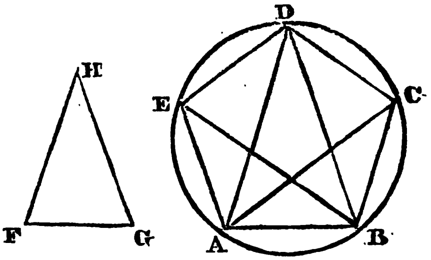
and therefore one tenth part of four right angles, or [Cor. 2. 15. 1], of all the angles which can be formed about the point A; whence, since in equal circles, equal angles at the centre, are subtended by equal chords (Cor. 2. 29. 3), BD, or its equal AC, is equal to the side of an equilateral, and therefore (Schol. 6. 4 and Def. 7. 4), of a regular decagon inscribed in the circle.

Cor. 2.—The square of the side of a regular pentagon inscribed in a circle, is equal to the squares of the radius and side of a regular decagon inscribed in the same.

Inscribe in the circle BDG, the right lines BF, FG each equal to BD (1. 4), draw AF, AG, GB, GC, and let fall the perpendicular GH on AB; and, since all the sides of the triangle AFB, are severally equal to all the sides of the triangle ABD, the angles FAB, BAD are equal (8. 1); in like manner the angles GAF, BAD may be proved equal; therefore the angle GAB is double the angle BAD, and one fifth part of four right angles; whence, in like manner, as in the preceding is shewn of BD, the right line GB may be shewn to be the side of a regular pentagon inscribed in the circle; and since, in the triangles GAC, ADB, the sides GA, AC and the angle GAC, are severally equal to the sides AD, DB and the angle ADB, the base GC is equal to AB (4. 1), or its equal GA, and so the triangle GAC is isosceles, and the perpendicular GH bisects AC (Cor. 26. 1); therefore, in the triangle BGA, the rectangle under the sum and difference of GB, GA, or, which is equal (Schol. 6. 2), the difference of their squares, is equal to the rectangle under the sum and difference of BH, HA (Cor. 1. 5 and 6. 2), or the rectangle ABC, or square of AC or BD; and so the square of GB the side of a regular pentagon inscribed in the circle, is equal to the squares of the radius AG, and of BD the side of a regular decagon inscribed in the same circle.

PROP. XI. PROB.

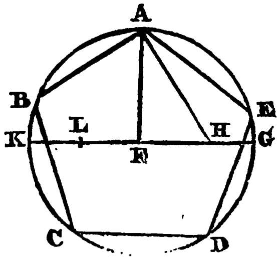
In a given circle (ABCDE), to inscribe a regular pentagon.



Make an isosceles triangle FGH, having each of the angles F, G double the angle H (10. 4), and inscribe in the given circle, the triangle ABD equiangular to the triangle FGH (2. 4), bisect the angles at the bases DAB, DBA by the right lines AC, BE, and join AE, ED, DC and CB.

Because the angles DAB, DBA, are each of them double to ADB, and bisected by the right lines AC, BE, the five angles ADB, DAC, CAB, ABE, EBD are equal, and therefore, the right lines AB, BC, CD, DE and EA are equal (Cor. 2. 29. 3) ; and so the pentagon ABCDE, which is inscribed in the given circle (Def. 3. 4), is equilateral, and, of course, regular (Schol. 6. 4).

Otherwise.



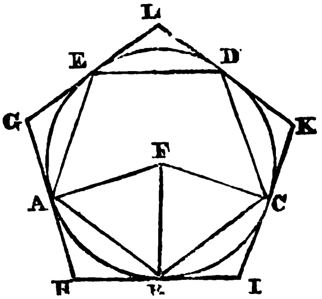
Draw two radiuses FA, FG, meeting each other at right angles in the centre F, divide FG in H, so that the rectangle FGH may be equal to the square of FH (11. 2), join AH, which is the side of the pentagon required.

For FH is equal to the side of a regular decagon, inscribed in the given circle (Cor. 1. 10. 4), and the square of AH is equal to the squares of AF and FH (47. 1), therefore AH is equal to the side of a regular pentagon inscribed in the same circle [Cor. 2. 10. 4] ; whence the required pentagon may be easily described, by applying in the given circle right lines AB, BC, CD and DE each equal to AH, and drawing AE.

Scholium.—The division of FG in H, as required above, may be performed, by producing GF to meet the circle in K, bisecting KF in L, and taking LH equal to the distance LA, as is manifest from the construction of Prop. 11. 2.

PROP. XII. PROB.

About a given circle (ABCDE), to circumscribe a regular pentagon.



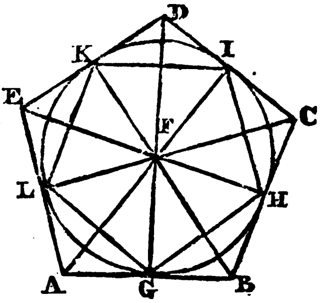
Inscribe in the given circle, the regular pentagon ABCDE [11. 4], through the vertices of the angles of which, draw GH, HI, IK, KL and LG, touching the circle in these vertices [17. 3]. The pentagon GHIKL, which is circumscribed about the given circle (Def. 5. 4), is a regular one.

For, drawing FA, FB and FC from the centre F, because the triangles FAB, FBC are mutually equilateral, the angles FAB, FBA, FBC, FCB are equal (8. 1), which being taken from the angles FAH, FBH, FBI, FCI, which are equal, being right [18. 3], the remaining angles HAB, HBA, IBC, ICB are equal; whence, in the triangles HAB, IBC, the right lines AB, BC being also equal (Constr. and Def. 7. 4), the sides AH, HB, BI, IC, as also the angles H and I are equal (26. 1); in like manner, AG may be proved equal to AH, HB or BI, therefore GH, HI are equal (Ax. 2). In like manner, may the sides IK, KL, LG be proved equal to GH or HI, and to each other; therefore the circumscribed pentagon is equilateral; it is also equiangular, since, in like manner as the angles H and I have been proved equal, the angles K, L, G may be proved equal to either of them, and to each other; it is therefore a regular one [Def. 7. 4].

Scholium.—In like manner, as is shewn in this proposition, may a regular polygon, be circumscribed about a given circle, of the same number of sides, as any given regular polygon, inscribed in the same circle.

PROP. XIII. PROB.

In a given regular pentagon (ABCDE), to inscribe a circle.



Bisect any two adjacent angles EAB, ABC, by the right lines AF, BF, meeting each other in F, and from F, draw FG perpendicular to AB; a circle, described from the centre F, at the distance FG, is inscribed in the given pentagon.

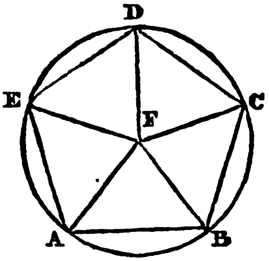
For let FC, FD, FE be joined, and on the sides of the pentagon, let fall the perpendiculars FH, FI, FK, FL.

Because then, in the triangles AFE, AFB, the sides AE, AB are equal [Hyp.], AF common, and the angles FAE, FAB equal [Constr.], the angles AEF, ABF are also equal [4. 1]; but the angles AED, ABC are equal [Hyp.], therefore, since ABF is the half of ABC [Constr.], AEF is half of AED; in like manner, it may be demonstrated, that the angles EDC, DCB are bisected by the right lines drawn to them; therefore, in the triangles FEL, FEK, the angles FEL, FEK are equal, as also the angles at L and K, being right angles (Constr.), and the side FE is common, therefore the sides FL, FK are equal [26. 1]; in like manner, it may be demonstrated, that all the other perpendiculars are equal to each other; therefore a circle described from the centre F, at the distance FG, passes through H, I, K, L, and, because of the right angles at G, H, I, K, L, touches the sides of the pentagon in these points [Cor. 16. 3], and is therefore inscribed therein (Def. 6. 4).

Schol.—In like manner, a circle may be inscribed, in any regular polygon.

PROP. XIV. PROB.

About a given regular pentagon (ABCDE), to circumscribe a circle.



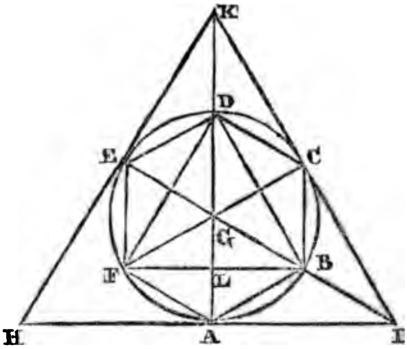
Bisect any two adjacent angles EAB, ABC by the right lines AF, BF meeting in F; a circle, described from the centre F, at the distance FA, is circumscribed about the given pentagon.

For let FB, FC, FD, FE be joined, and in the triangles FAB, FAE, the sides FA, AB and the angle FAB, are severally equal to FA, AE and the angle FAE, therefore the angles ABF, AEF are equal (4. 1); but the angles ABC, AED are equal (4. 1); but the angles ABC, AED are equal (Hyp. and Def. 7. 4), and ABF is the half of ABC, therefore AEF is the half of AED, and so the angle AED is bisected by EF; in like manner it may be shewn, that the other angles of the pentagon at D and C are bisected. Since then, in the triangle FAB, the angles FAB, FBA, being the halves of the equal angles EAB, ABC, are equal (Ax. 7), FA is equal to FB (6. 1); in like manner it may be shewn, that FC, FD, FE are each of them equal to FA or FB, therefore the five right lines FA, FB, FC, FD and FE are equal to each other, and the circle, described from the centre F, at the distance FA, passes through B, C, D and E, and is circumscribed about the given pentagon (Def. 4. 4).

Schol.—In like manner, a circle may be circumscribed, about any regular polygon.

PROP. XV. PROB.

In a given circle (ABCDEF), to inscribe a regular hexagon.



Having found the centre G of the given circle, and drawn the diameter AGD, inscribe in the circle AB, DC, AF, DE each equal to AG, join GB, GC, GE, GF, then since the triangles AGB, CGD, DGE, FGA are equilateral ones, and therefore equiangular (Cor. 5. 1), their angles at the point G, are each of them equal to one third of two right angles (32. 1), and the angles AGB, DGC together, equal to two third parts of two right angles, and therefore the angle BGC equal to a third part of two right angles (Cor. 13. 1); in like manner it may be shewn, that the angle EGF is a third part of two right angles, and so each of the six angles, formed at the point G, is equal to a third part of two right angles, and therefore to each other; whence, the chords subtending them, which are the sides of the hexagon ABCDEF, inscribed in the circle (Def. 3. 4), are equal (Cor. 2. 29. 3), which hexagon is therefore regular (Schol. 6. 4. and Def. 7. 4).

Cor. 1.—The side of a regular hexagon, inscribed in a circle, is equal to the radius.

For, in the above construction, AB one of the sides of the inscribed hexagon, is equal to the radius AG (Constr.).

Cor. 2.—The square of the side of a regular pentagon, inscribed in a circle, is equal to the squares, of the sides of a regular hexagon, and regular decagon, inscribed in the same circle.

For the square of the side of the pentagon, is equal to the squares of the radius and the side of the decagon (Cor. 2. 10. 4), or, radius being equal to a side of the hexagon (Cor. 1. 15. 4), to the squares of the sides of the hexagon and decagon.

Cor. 3.—The square of the side of an equilateral triangle, is triple the square of the radius of the circumscribing circle.

For FB, BD, DF being joined, are equal, because they subtend equal angles FGB, BGD, DGF at the centre (Cor. 2. 29. 3), and so the triangle FBD is equilateral, and the angle DBA in a semicircle is right (31. 3), therefore the squares of DB, BA together, are equal to the square of DA (47. 1), or, which is equal (Cor. 4. 2), to four times the square of AG, or (Constr.), AB; taking from each the square of AB, there remains the square of DB, a side of the equilateral triangle FBD inscribed in the circle, equal to three times the square of AB, or, of the radius AG.

Cor. 4.—The altitude of an equilateral triangle, is triple the radius of the inscribed, and triple to half the radius of the circumscribed circle.

Part 1.—Let the right lines HAI *(see fig. on preced. page)*, ICK and KEH touching the circle ACE in A, C and E, meet each other in H, I and K; join DK, BI, and in the triangles DCK, DEK, the sides DC, DE are equal, being sides of the hexagon, CK is equal to EK (Cor. 2. 36. 3), and DK is common, therefore the angles CDK, EDK are equal (8. 1); whence the angles GDC, GDE being angles of equilateral triangles, and therefore equal, the angles GDC, CDK together are half the four angles GDC, CDK, KDE, EDG, and therefore equal to two right angles (Cor. 2. 15. 1), therefore AD, DK make one right line (14. 1); and since the angle DCK is equal to the angle in the alternate segment DFC (32. 3), or to CDB, which is equal to DFC (27. 3), they standing on the equal arches DC, CB, the right lines CK, BD are parallel (27. 1); whence, the angle DCK being equal to DFC in the alternate segment (32. 3), or to BDA, which is equal to DFC (27. 3), they standing on equal arches DC, AB, or, which is equal (29. 1), the internal remote angle CKD, DK is equal to DC (6. 1), or the radius AG; in like manner, EB, BI may be proved to make one right line, and BI to be equal to AG; and since, in the triangles GCI, GCK, the angles at G are equal, those at C right, and GC common, CI, CK are equal (26. 1), and CK, EK are equal (Cor. 2. 36. 3); in like manner may EH, HA and AI be each of them proved equal to CI, CK or EK, and to each other; therefore HI, IK and KH are equal to each other (Ax. 2 or 6), and the triangle HIK equilateral (Def. 28. 1), and circumscribed about the circle ACE (Def. 5. 4); of which triangle, AK, being at right angles to HI (18. 3), is the altitude, which, DK being equal to AG or GD, is triple the radius AG.

Part 2.—In the triangles ABL, GBL, AB is equal to BG, BL common, and the angles ABL, GBL standing on the equal arches AF, FE also equal (27. 3), therefore AL and GL are equal, and the angles ALB, GLB are equal (4. 1), and therefore right angles [Def. 20. 1]; whence, FBD being an equilateral triangle, inscribed in the circle ACE, as appears by the preceding corollary, and, because of the right angles at L, DL being its altitude, that altitude is triple the half radius AL or GL.

Cor. 5.—Of two regular polygons, of an unequal number of sides, an angle of that which has the greater number of sides, is greater, than an angle of the other.

First, let the difference of the number of sides be one, and an angle of that, which has the greater number, is not equal to an angle of the other, for then the total of its angles, would exceed the total of the angles of the other, only by one of the angles of that other, and, therefore, by less than two right angles, which is absurd [Cor. 1. 32. 1]. A like absurdity would follow, if an angle of that, which has the greater number of sides, were supposed to be less; since therefore it can neither be equal to, or less than, an angle of that other, it is greater.

Secondly, let the difference of the number of sides be two; and an angle of that, which has the greater number of sides, is greater, than an angle of that, which has the next greater (Part 1. of this Cor.), and an angle of that, than an angle of the other (Part 1. of this Cor.).

In like manner we might proceed, if the difference of the number of sides were three, four, or any other number.

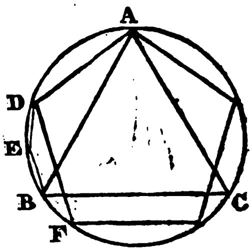
Cor. 6.—Only three regular figures, namely, equilateral triangles, squares and hexagons, can constitute one continued superficies.

For, in order thereto, it is necessary, that a certain number, of the angles of the regular figure, which is to constitute such a superficies, should be equal to four right angles, all the angles which can be formed about any point being equal to so many right ones [Cor. 2. 15. 1]; but six angles of an equilateral triangle are equal to four right angles, three of them being equal to two right ones (32. 1), the four angles of a square are four right angles [Def. 36. 1), and, since all the six angles of a regular hexagon, are equal to eight right angles (Cor. 1. 32. 1), three of them are equal to four right ones.

But the angles, of no other regular figure, can constitute a continued superficies, for no fewer angles, of a regular polygon, than three, can be equal to four right angles, since no such angle can be equal to two right angles, for then the two sides would be in the same right line; and because the greater the number of sides of a regular polygon, the greater is each of its angles [Cor. 5. of this prop.], the three angles of a regular pentagon are less than the three angles of a regular hexagon, which are shewn above to be equal to four right angles; and, for the same reason, four angles of a regular pentagon are greater than the four angles of a square, or four right angles; and much more, is a greater number of such angles, greater than four right ones. Also, the three angles of a regular heptagon, or seven sided polygon, and much more of any regular polygon of a greater number of sides, are greater than the three angles of a regular hexagon, or four right angles.

PROP. XVI. PROB.

In a given circle (ABD), to inscribe a regular quindecagon.



Let AB be the side of an equilateral triangle ABC, inscribed in the circle [1. 1 and 2. 4], and AD, the side of a regular pentagon, inscribed in the same (11. 4), bisect the arch BD in E (30. 3), and draw BE, DE.

And since, of such equal parts, as the whole circumference ADCA contains fifteen, the arch ADB, being the third part of the whole, contains five, and the arch AD, the fifth part, contains three, their difference BD contains two such parts, and BE or ED one each; and so the right lines BE, ED are sides of a regular quindecagon, inscribed in the given circle; which is, of course, formed, by inscribing around in the circle, right lines equal to either of these right lines [1. 4].

BOOK V.

DEFINITIONS.

1. Of two unequal magnitudes of the same kind, the greater is said to be a *multiple* of the less; when the less measures the greater, or, being repeated any number of times, the aggregate is equal to the greater.

Two magnitudes, are said to be *equimultiples*, or *like multiples*, of two others, each of each; when the two latter being repeated an equal number of times, become severally equal to the two former.

One magnitude, is said to be *sesquialteral* of another, when the former is equal to one and a half of the latter.

2. One magnitude, is said to be a *submultiple* of another, the less of the greater; when the less measures the greater.

Two magnitudes, are said to be *equisubmultiples* of two others, each of each; when the latter are equimultiples of the former.

3. *Ratio*, is a certain relation, between two magnitudes of the same kind, with respect to quantity.

Of which two magnitudes or terms, the first, or magnitude referred, is called the *antecedent*, and that to which it is referred, the *consequent*.

4. Magnitudes, are said to *have ratio* to each other, when the less may be so multiplied, as to exceed the greater. *(See note to this definition).*

5. The first, of four magnitudes, is said to have the *same* (or an *equal*) *ratio* to the second, as the third has to the fourth; when any equisubmultiples whatever of the second and fourth, are contained equally often in the first and third respectively.

6. Magnitudes, which are in the same ratio, are called *proportionals*.

When four magnitudes are proportionals, the first is said to be to the second, as the third is to the fourth.

7. But the first, of four magnitudes, is said to have a greater ratio to the second, than the third has to the fourth; when, of any equisubmultiples whatever of the second and fourth, the submultiple of the second is contained oftener in the first, than that of the fourth is in the third.

Cor.—From this and the fifth definition of this book, it follows, that the ratio of two magnitudes to each other, cannot be both equal and unequal to another ratio, since, if any equisubmultiples whatever of the consequents, be continued an unequal number of times in the antecedents, the ratios are not equal [Def. 5, 5].

8.—*Proportion,* is a similitude of Ratios.

Proportion cannot consist in less than three terms.

9. Three magnitudes, are said to be proportional, when the first has the same ratio to the second, as the second has to the third.

10. Of three proportional magnitudes, the middle one, is said to be, a *mean proportional,* between the other two; and the last, a *third proportional,* to the first and second.

11. Of four proportional magnitudes, the last is said to be a *fourth proportional,* to the other three taken in order.

12. Magnitudes, are said to be *continually proportional,* when the first has the same ratio to the second, as the second to the third, as the third to the fourth, and so on..

13. The ratio of the first, of any number of magnitudes of the same kind, to the last, is said to be compounded, of the ratios of the first to the second, the second to the third, and so on to the last.

14. In a series of magnitudes continually proportional, the ratio of the first to the third, is said to be *duplicate* to that of the first to the second; the ratio of the first to the fourth, *triplicate* to that of the first to the second; and so on. Also the ratio of the first, to a mean proportional between the second and third, to be *sesquiplicate* to that of the first to the second. But the first is said to have to the second, a *subduplicate ratio* of the first to the third, a *subtriplicate* of the first to the fourth, and so on.

15. In proportionals, the antecedents are said to bo *homologous* (or co-rational) terms, as also the consequents.

The order or magnitude of proportionals may be so changed, that they may nevertheless be still proportionals, in various manners; which, among geometers, are designated by the following names.

16. *Alternating;* when it is inferred, if four magnitudes of the same kind be proportional, that the first is to the third,

as the second is to the fourth; as is shewn in the 16 prop. of this book.

17. *Inverting;* when it is inferred, if four magnitudes be proportional, that the second is to the first, as the fourth to the third; as is shewn in Theor. 3 at 15th of this book.

18. *Compounding;* when it is inferred, if four magnitudes be proportional, that the first and second together is to the second, as the third and fourth together to the fourth. Prop. 18th of this book.

19. *Dividing;* when it is inferred, if four magnitudes be proportional, that the difference between the first and second is to the second, as the difference between the third and fourth is to the fourth. Prop. 17th of this book and scholium thereto.

20. *Converting;* when it is inferred, if four magnitudes be proportional, that the first is to the sum or difference of it and the second, as the third is to the sum or difference of it and the fourth. See Schol. to prop. 18th of this book.

21. *Equality;* when it is inferred, if there be more than two magnitudes, and others equal to them in number, which, being taken two and two, are in the same ratio; that the first is to the last in the former series, as the first to the last in the latter.

Of this, there are the two following species.

22 *Ordinate Equality;* when the first magnitude is to the second in the former series, as the first to the second in the latter, and the second to the third in the former, as the second to the third in the latter, and so on; and it is inferred, as in the preceding definition, that the first is to the last in the former series, as the first to the last in the latter. Prop. 22. 5.

23. *Perturbate Equality;* when the first magnitude is to the second in the former series, as the last but one to the last in the latter, and the second to the third in the former series, as the last but two to the last but one in the latter, and so on; and it is inferred, as in the 21st definition, that the first is to the last in the former series, as the first to the last in the latter. Prop. 23. 5.

POSTULATES.

1. That a magnitude may be taken equal to any given magnitude, or to the half, third or fourth, &c. thereof.

2. That of any two unequal magnitudes of the same kind, such a multiple of the less may be taken, as to exceed the greater.

AXIOMS.

1. Equimultiples of the same, or equal things, are equal.

2. Equisubmultiples of the same or equal things, are equal.

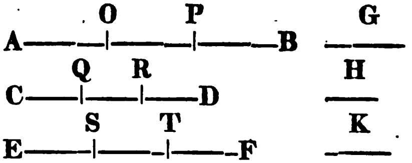
Cor. 1.—Equimultiples of unequal magnitudes are unequal, that of the greater being the greater.

Cor. 2.—Equisubmultiples of unequal magnitudes are unequal, that of the greater being the greater.

Cor. 3.—If two equal magnitudes be less than two other equal magnitudes, the former are contained equally often in the latter.

PROPOSITION I. THEOREM.

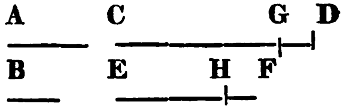
If any number of magnitudes of the same kind (AB, CD, EF), be equimultiples of a like number of magnitudes (G, H, K), each of each; whatever multiple any one of the former magnitudes (as AB), is of its correspondent one (G), a like multiple are all the former together, of all the latter together.



For, because AB, CD, EF are equimultiples of G, H, K, there are in AB, as many magnitudes equal to G, as there are in CD, equal to H, and EF equal to K; let AB be divided into parts AO, OP, PB, each equal to G [Post. 1. 5], CD into parts CQ, QR, RD, each equal to H (by the same), and EF into parts ES, ST, TF each equal K (by the same); and, because AO is equal to G, CQ to H, and ES to K; AO, CQ and ES together, are equal to G, H and K together [Ax. 2. 1]; for the same reason, OP, QR and ST together, as also PB, RD and TF together, are equal to G, H and K together; therefore whatever multiple AB is of G, CD of H, or EF of K, a like multiple are AB, CD, EF together, of G, H, K together.

Cor. 1.—If any number of magnitudes of the same kind, be contained equally often in the same number of magnitudes, each in each; as often as any one of the former magnitudes, is contained in its corresponding one among the latter, so often is the compound of all the former, in the compound of all the latter.

If all the former be submultiples of all the latter, it is manifest from this proposition.



But if two magnitudes A, B, be contained equally often in CD and EF, neither of them being submultiples of its corresponding one; let A and B be taken from those which correspond to them as often as possible, there being left of CD, a magnitude GD less than A, and of EF, HF less than B; CG and EH are equimultiples of A and B (Hyp. and Constr.), therefore CG and EH together, is a like multiple of A and B together, as CG is of A (1. 5); therefore A and B together, is contained equally often in CG and EH together as A is in CG or CD; but because A is greater than GD, and B than HF, A and B together, is greater than GD and HF together; therefore A and B together, is not oftener contained in CD and EF together, than in CG and EH together; and therefore A and B together, is contained equally often in CD and EF together, as A is in CD.

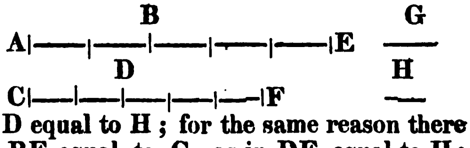
In like manner, it may be demonstrated, if A be a submultiple of CD, and B not a submultiple of EF.

And in like manner, it may be demonstrated, if there be ever so many magnitudes.

Cor. 2.—As often as any magnitude, is contained in another, so often is any multiple of the former, contained in a like multiple of the latter. This being but a case of the preceding corollary.

PROP. II. THEOR.

If two magnitudes (AB, CD) be equimultiples of two others (G, H), each of each, and to the former, there be added, equals or equimultiples (BE, DF) of the latter (G, H); the compounds (AE, CF), are equimultiples of the latter.

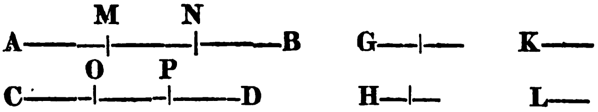


For, because AB, CD are equimultiples of G, H, there are as many magnitudes in AB equal to G, as in CD equal to H; for the same reason there as many magnitudes in BE equal to G, as in DF equal to H; therefore there are as many magnitudes in AB and BE together, or, AE, equal to G, as in CD and DF together, or CF, equal to H [Ax. 2. 1).

Cor—In like manner it may be proved, that, if two magnitudes (G, H), be contained equally often in two others (AB, CD), each in each, and to the latter (AB, CD) there be added, equals or equimultiples (BE, DF) of the former (G, H); the same former (G, H), are contained equally often in the compounds (AE, CF).

PROP. III. THEOR.

If two magnitudes (AB, CD), be equimultiples of two others (G, H), each of each; they are also equimultiples of any like submultiples (K, L) thereof.



Because AB, CD are equimultiples of G, H (Hyp.), there are as many parts in AB equal to G, as in CD equal to H; let AB be divided into parts AM, MN, NB each equal to G, and CD into parts CO, OP, PD each equal to H (Post. 1. 5); and, since G, H are equimultiples of K, L, and AM equal to G, and CO to H, AM and CO are equimultiples of K, L; for a like reason, MN, OP are equimultiples of K L; therefore AN, CP are also equimultiples of K, L (2. 5). In like manner may AB, CD be proved to be equimultiples of K, L.

Cor. 1.—If two magnitudes (K, L, see the above fig.), be contained an unequal number of times in two others (G, H; K being contained oftener in G, than L in H); they are contained an unequal number of times in any equimultiples thereof (AB, CD.

Since K is contained oftener in G, than L is in H, it is also contained oftener in AM, MN, NB, which are, each of them, equal to G, than L is in CO, OP, PD, which are, each of them, equal to H; and therefore K is contained oftener in the whole AB, than L is in the whole CD (Ax. 4. 1).

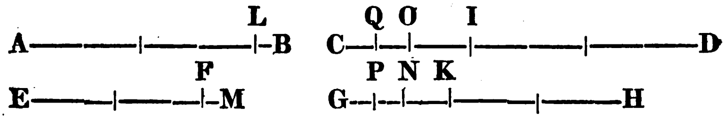
Cor. 2.—If any two magnitudes (K, L), be contained equally often in two others (AB, CD), each in each; any equimultiples (G, H) of them, are contained equally often in the same.

For, if one of them, as G, were contained oftener in AB, than H in CD; K would (by the preced. Cor.) be contained oftener in AB, than L in CD, contrary to the supposition.

Cor. 3.—If two magnitudes (K, L), be contained equally often in two others (AB, CD); they are contained equally often in any equisubmultiples thereof (G. H).

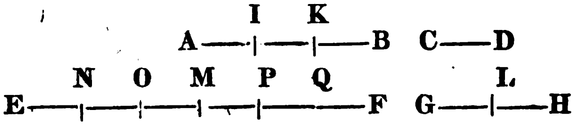
For, if one of them K, were contained oftener in G, than L is in H; K would be contained oftener in AB, than L in CD (Cor. 1. 3. 5), contrary to the supposition.

Cor. 4.—A ratio (as of AB to CD) cannot be both greater and less than another (as of EF to GH). In other words, if a submultiple CI of CD be contained oftener in AB, than a like submultiple GK of GH is in EF; no submultiple of CD is contained less often in AB, than a like submultiple of GH is in EF. See Def. 7. 5.



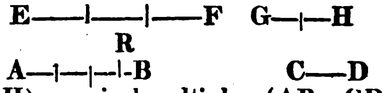
If possible, let CI be contained oftener in AB, than CK in EF, and let AL be the greatest multiple of CI, which is in AB, and EM a like multiple of GK, which is greater than EF; and, if possible, let GN a submultiple of GH, be contained oftener in EF, than CO a like submultiple of CD, is in AB, and let CQ, GP be like submultiples of CO, GN, as CI, GK are of AL, EM; and since GN is contained oftener in EF, and therefore in EM, than CO is in AB, and therefore than CO is in AL; and CQ, GP are contained equally often in CI, GK, as CO, GN are in AL, EM (Cor. 2. 1. 5); therefore GP is contained oftener in GK, than CQ is in CI; and therefore, GH, CD being equimultiples of GK, CI, GP is contained oftener in GH, than CQ in CD (Cor. 1. 3. 5), which is absurd, CQ, GP being equisubmultiples of CO, GN, and therefore of their equimultiples CD, GH (3. 5); therefore GN is not contained oftener in EF, than CO is in AB: in like manner it may be proved, that no submultiple of GH, is contained oftener in EF, than a like multiple of CD, is in AB.

Theor. 1.—*(See note.)* If, of two magnitudes (AB, CD), one of which (AB) is a multiple of the other (CD), equimultiples (EF, GH) be taken; the multiple (EF) of the former, is a like multiple of that (GH) of the latter, as the former (AB), is of the latter (CD).



Let AB be divided into parts AI, IK, KB each equal to CD, GH into parts GL, LH each equal to CD, EF into parts EM, MF each equal to AB, and EM, MF into parts EN, NO, OM, MP, PQ, QF each equal to KB or CD; and it is manifest, by taking on EF, like multiples EO, OP, PF of CD, as GH is of CD, that there are as many parts in EF equal to GH, as in AB equal to CD, and therefore that EF is a like multiple of GH, as AB is of CD.

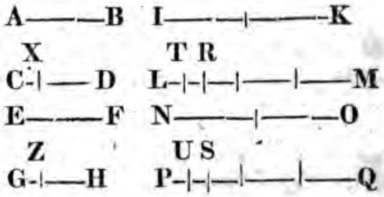
Theor. 2.—If, of two magnitudes [EF, GH], one of which (EF) is a multiple of the other (GH), equisubmultiples (AB, CD) be taken; the submultiple [AB] of the former, is a like multiple of that (CD) of the latter, as the former (EF), is of the latter (GH).



If AB be not a like multiple of CD, as EF is of GH, it is either greater or less than such a multiple; let it, if possible, be greater, and take on it AR equal to such a multiple; and since GH is a multiple of CD [Hyp.], and EF, AR equimultiples of GH, CD, by the preced. Theor. EF is a like multiple of AR, as GH is of CD; but EF is a like multiple of AB, as GH is of CD [Hyp.]; therefore EF is a like multiple of AR as of AB; whence AR and AB being equisubmultiples of EF are equal [Ax. 2. 5], part and whole, which is absurd. A like absurdity would follow, if AB were supposed to be less than a like multiple of CD, as EF is of GH; since therefore AB is neither greater or less than such a multiple, it is such a multiple.

PROP. IV. THEOR.

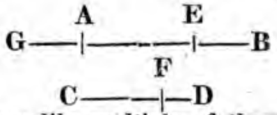
If there be four proportional magnitudes (AB, CD, EF, GH), and equimultiples (IK, NO) of the antecedents (AB, EF) be taken and equimultiples (LM, PQ) of the consequents (CD, GH); these are also proportional.



Let LR, PS be any equisubmultiples whatever of LM, PQ, and take LT, PU like submultiples of LR, PS, as AB is of IK, and CX, GZ like submultiples of CD, GH, as LT, PU are of LM, PQ; and since LM is a multiple of CD [Hyp.], and LT, CX equisubmultiples of LM, CD; LT is a like multiple of CX, as LM of CD (Theor. 2 at 3. 5); in like manner it may be shewn, that PU is a like multiple of GZ, as PQ of GH; therefore LT, PU are equimultiples of CX, GZ; and CX, GZ are contained equally often in AB, EF (Hyp. and Def. 5. 5), therefore LT, PU are also contained equally often in AB, EF (Cor. 2. 3. 5); and LR, PS are contained equally often in IK, NO, as LT, PU in AB, EF (Cor. 2. 1. 5), therefore LR, PS, which are any submultiples whatever of LM, PQ, are contained equally often in IK, NO; therefore IK is to LM, as NO to PQ [Def. 5. 5].

PROP. V. THEOR.

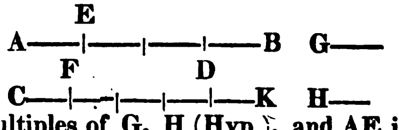
If a magnitude (AB), be a like multiple of another (CD), as a part (AE) taken from the first, is of a part (CF) taken from the other; the remainder of the first (EB), is a like multiple of the remainder of the other (FD), as the whole (AB), is of the whole (CD).



Take AG a like multiple of FD, as AE is of CF; then GE is a like multiple of CD, as AE of CF (1. 5), or (Hyp.), as AB of CD; whence, GE, AB being equimultiples of CD are equal (Ax. 1. 5), taking from each AE common, GA is equal to EB (Ax. 3. 1): but GA is a like multiple of FD, as AE is of CF; therefore EB is a like multiple of FD, as AE is of CF, or (Hyp.), as AB is of CD.

PROP. VI. PROB.

If two magnitudes (AB, CD), be equimultiples of two others (G, H), and from each of the former, there be taken away equimultiples (EB, FD) of the latter, the remainders (AE, CF) are either equals or equimultiples of the latter.



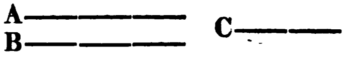
For, first, let AE be equal to G; CF is equal to H. Let DK be taken to H (Post. 1. 5), and, because EB, FD are equimultiples of G, H (Hyp.), and AE is equal to G, and DK to H, AB, FK are equimultiples of G, H (2. 5); but AB, CD are also equimultiples of G, H [Hyp.], therefore CD, FK are equimultiples of H, and therefore equal [Ax. 1. 5]; taking away FD common, CF is equal to DK [Ax. 3. 1], and therefore to its equal H.

In like manner, if AE were a multiple of G, CF might be shewn to be a like multiple of H.

Cor.—In like manner it may be proved, that if two magnitudes (G, H), be contained equally often in two others (AB, CD), and from the latter, there be taken equimultiples (EB, FD) of the former, the former are contained equally often in the residues (AE, CF).

PROP. VII. THEOR.

Equal magnitudes (A, B), have the same ratio to the same magnitude (C): and the same magnitude (C), has the same ratio to equal magnitudes (A, B).

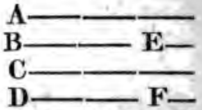


1. Since any submultiple whatever of C, is contained equally often in the equal magnitudes A, B (Cor. 3. Ax. B. 5), the ratios of A to C and of B to C are equal (Def. 5. 5).

2. Since any equisubmultiples whatever of the equal magnitudes A, B are equal (Ax. 2. 5), they are contained equally often in the same magnitude C (Cor. 3. Ax. B. 5); therefore the ratios of C to A and of C to B are equal [Def. 5. 5].

Schol.—In like manner it may be shewn, that "equal magnitudes have the same ratio to equal magnitudes."

Cor. 1.—If the first (A) of four magnitudes of the same kind, be equal to the third (C), and the second (B), to the fourth (D); the magnitudes are proportional.



Take any equimultiples whatever E and F of B and D; E and F are equal (Hyp. and Ax. 2. 5), and are therefore contained equally often in the equals A and C (Cor. 3. Ax. B. 5), therefore A is to B, as C is to D (Def. 5. 5).

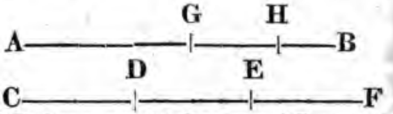
Cor. 2.—If the first of four magnitudes be equal to the second, and the third to the fourth; the magnitudes are proportional.

For any submultiples whatever of the consequents, are like submultiples of the antecedents, and therefore contained equally often in them; therefore the four magnitudes are proportional [Def. 5. 5].

Theorem, which is prop. 1 *of B.* 10 *of Euclid.*

If from the greater [AB], of two unequal magnitudes [AB, CD], there be taken away more than the half, and from the remainder [GB], be again taken away more than the half, and and so on continually; there would at length be left a magnitude (HB), less than the least [CD], of the two magnitudes first proposed.

Let magnitudes DE, EF be taken each equal to CD [Post. 1. 5], until a multiple CF of CD, be greater than AB [Post. 2. 5]; let AG be a part taken on AB greater than its half, and GH a part on GB greater than its half, and so on, till there be the same number of parts in AB, as in CF; the last remainder HB is less than CD.

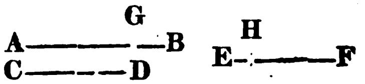


For AB being less than CF, and from AB, AG being taken greater than the half, and from CF, CD not greater than the half, being less than the half, unless CF be only double to CD, in which case it is equal to the half, the remainder GB is less than the remainder DF; in like manner, HB may be proved to be less than EF or CD.

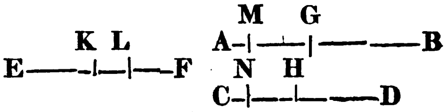
Cor.—The same thing may be proved, and in the same manner, if from AB should be taken but the half [Post. 1. 5], and from the remainder its half, and so on; in which case HB would be a submultiple of AB [3 of this]; and therefore, "a submultiple may be taken of any given magnitude, less than any other given one of the same kind."

PROP. VIII. THEOR.

The greater (AB), of two unequal magnitudes (AB, CD), has the greater ratio to the same magnitude (EF); and the same magnitude (EF), has the greater ratio to the less (CD).



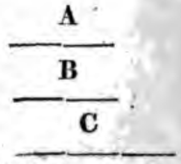
Part 1.—Let AG be a part taken on AB equal to CD [Post 1. 5], and GB be the excess of AB above CD; let EH be a submultiple of EF less than GB [Cor. Theor. at 7. 5], and EH is contained as often in AG as CD, and being less than GB is contained at least once therein, it is therefore oftener contained in AB than in CD, and therefore the ratio of AB to EF is greater than that of CD to EF [Def. 7. 5].



Part 2.—Let AG be a submultiple of AB less than EF [Cor. Theor. at 7. 5], and CH a like submultiple of CD [Post. 1. 5]; CH is less than AG [Hyp. and Cor. 2. Ax. B. 5], and therefore is either oftener contained in EF, or being contained equally often therein leaves a greater remainder [Cor. 1. Ax. B. 5 and Ax. 5. 1], if it be oftener contained therein, the proposition is true [by Def. 7. 5]; if not, being contained equally often therein, let it leave a remainder KF, greater than that LF, left by the multiple of AG, and let KL be the difference of these remainders; let CN be a submultiple of CH less than KL [Cor. Theor. at 7. 5], and AM a like submultiple of AG [Post. 1. 5], and AM, CN being like submultiples of AG, CH, are like submultiples of their equimultiples EL, EK (3. 5), whence CN, which is less than AM [Cor. 2. Ax. B. 5], being at least as often contained in LF, as AM is in the same LF [Cor. 1. Ax. B. 5], CN is contained at least as often in the compound of EK and LF, as AM is in the compound of EL and LF, or in the whole EF; but CN, being less than KL, is contained at least once therein, and is therefore contained oftener in EF, than in the compound of EK and LF; but it has been shewn, that CN is contained as often in the compound of EK and LF, as AM is in EF; therefore CN is contained oftener in EF, than AM in the same EF; whence, CN, AM being like submultiples of CH, AG, and therefore of their equimultiples CD, AB (3. 5), the ratio of EF to CD is greater than that of EF to AB [Def. 7. 5].

PROP. IX. THEOR.

Magnitudes (A, B), which have the same ratio to the same magnitude (C), are equal: And those magnitudes (A, B , to which the same magnitude (C) has the same ratio, are equal.

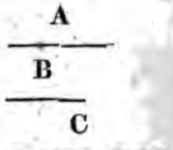


Part 1.—If A and B be not equal, let one them, if possible, as A, be the greater ; and the ratio of A to C is greater than that of B to C [8. 5], which is absurd [Hyp. and Cor. Def. 7. 5] ; therefore A and B are not unequal, they are of course equal.

Part 2.—If A and B be not equal, let one of them, if possible, as B, be the less ; and the ratio of C to B is greater than that of C to A [8. 5], which is absurd [Hyp. and Cor. Def. 7. 5] ; therefore A and B are not unequal, they are of course equal.

PROP. X. THEOR.

That (A), of two magnitudes (A, B), which has the greater ratio to the same magnitude (C), is the greater; and that (B), to which the same magnitude (C) has the greater ratio, is the less.

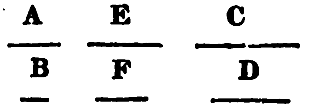


Part 1.—A is not equal to B, for if it were, its ratio to C would be the same, as that of B to C (7. 5), contrary to the supposition and Cor. Def. 7. 5; A is not less than B, for then B the greater, would have the greater ratio to C (8. 5), contrary to the supposition and Cor. 4. 3. 5 ; therefore A is greater than B.

Part 2.—B is not equal to A, for then C would have the same ratio to B as to A [7. 5], contrary to the supposition and Cor. Def. 7. 5 ; B is not greater than A, for then C would have a greater ratio to the less A, than to the greater B [8. 5], contrary to the supposition and Cor. 4. 3. 5 ; therefore B is less than A.

PROP. XI. THEOR.

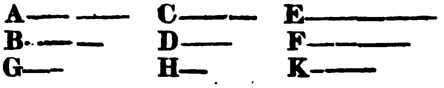
Ratios (as of A to B and C to D), which are the same to the same ratio (as of E to F), are the same to each other.



For, since any equisubmultiples whatever of B and D, are contained in their respective antecedents A and C, the same number of times, as a like submultiple of F is in E [Hyp. and Def. 5. 5]; it follows [Ax. 1. 1], that any equisubmultiples whatever of B and D are contained equally often in the same antecedents A and C; therefore A is to B, as C is to D [Def. 5. 5].

PROP. XII. THEOR.

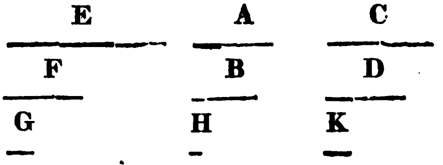
If any number of magnitudes of the same kind (A, C, E), have the same ratio to the same number of magnitudes (B, D, F), each to each; one of the antecedents (A) is to its consequent (B), as all the antecedents (A, C, E) together, are to all the consequents (B, D, F) together.



Let G, H, K be any equisubmultiples whatever of B, D, F [Post. 1. 5], and G, H, K together, is a like submultiple of B, D, F together, as one of them G, is of its correspondent magnitude B [1. 5]; also, since G, H, K are contained equally often in A, C, E respectively [Hyp. and Def. 5. 5], the aggregate of G, H, K, is contained in the aggregate of A, C, E, the same number of times, as any one of them G, is in the correspondent and magnitude A [Cor. 1. 5]; and, this being true, whatever submultiples G, H, K are of B, D, F, any one of the antecedents A, is to its consequent B, as all the antecedents A, C, E together to all the consequents B, D, F together [Def. 5. 5].

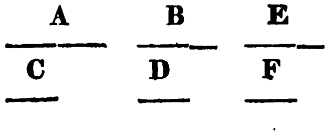
PROP. XIII. THEOR.

Any ratio (as of E to F), which is greater than one (as of A to B), of two equal ratios (of A to B and of C to D), is also greater than the other (namely, that of C to D).



Since the ratio of E to F is greater, than that of A to B [Hyp.], a submultiple of F, as G, may be taken, which is contained oftener in E, than a like submultiple H of B is in A [Def. 7. 5); take K a like submultiple of D, as H is of B [Post. 1. 5], and H and K are contained equally often in A and C [Hyp. and Def. 5. 5], but G is contained oftener in E, than H is in A, therefore G is contained oftener in E, than K is in C, and therefore, G, K being equisubmultiples of F, D, the ratio of E to F is greater than that of C to D [Def. 7. 5].

Schol.—In like manner, if, in this proposition, the ratio of E to F were supposed to be less than of A to B, that of A to B being equal to that of C to D, the ratio of E to F may be shewn to be less than that of C to D.—And if the ratio of E to F being supposed greater than of A to B, that of A to B were supposed to be greater than of C to D: it might, in like manner, be proved that the ratio of E to F is greater than of C to D: for, a submultiple of F being taken, which is contained oftener in E, than a like submultiple of B is in A [Hyp. and Def. 7. 5], the submultiple of B is not contained less often in A, than a like submultiple of D in C (Cor. 4. 3. 5), therefore the submultiple of F is contained oftener in E, than that of D in C, and therefore the ratio of E to F is greater than that of C to D (Def. 7. 5). Therefore " a ratio (as of E to F) which is greater than another " (as of A to B), is greater than any ratio [as of C to D] less " than that other."



Cor.—If the first (A) of four proportionals, be greater than the second (B), the third (C), is greater than the fourth (D), if equal, equal, and if less, less.

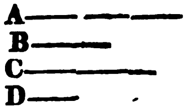
First, let A be greater than B, and take E equal to B, and F to D. The ratio of C to D is equal to that of A to B (Hyp.), or (Hyp. and 8 and 13. 5), greater than that of E to B, or

(Cor. 2. 7. 5), than that of F to D; since therefore the ratio of C to D is greater than that of F to D, C is greater than F (10. 5), or its equal D.

In like manner it might be proved, that, if A were equal to B, C would be equal to D, and if less, less.

PROP. XIV. THEOR.

If the first (A), of four proportional magnitudes (A, B, C, D), be greater than the third (C), the second (B), is greater than the fourth (D), if equal, equal, and if less, less.



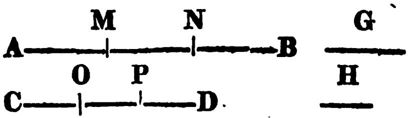
First, let A be greater than C; B is greater than D.

For, because A is greater than C, the ratio of A to B is greater than that of C to B (8. 5), but A is to B as C to D (Hyp.), therefore the ratio of C to D is greater than that of C to B (13. 5), therefore B is greater than D (10. 5).

In like manner, if A were equal to C, B may be proved to be equal to D, and if less, less.

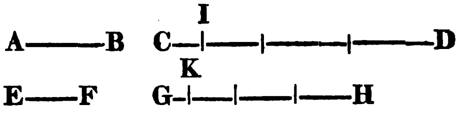
PROP. XV. THEOR.

Magnitudes (G, H) have the same ratio to each other, as their equimultiples (AB, CD).



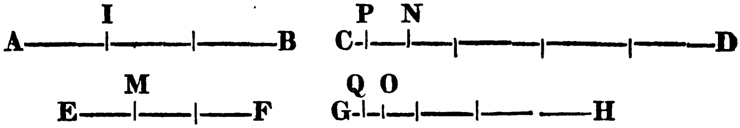
Because AB, CD are equimultiples of G, H, there are as many magnitudes in AB equal to G, as in CD equal to H; let AB be divided into parts AM, MN, NB each equal to G, and CD into parts, CO, OP, PD each equal to H [Post. 1. 5]; and since AM, MN, NB are equal, as also CO, OP, PD, AM is to CO, as MN to OP, and as NB to PD [Cor. 1. 7. 5], therefore AM, MN, NB together, or AB, is to CO, OP, PD together, or CD, as AM is to CO [12. 5], or [Cor. 1. 7. 5], as G is to H.

Theor. 1.—Magnitudes (AB, EF), have the same ratio to their equimultiples (CD, GH).



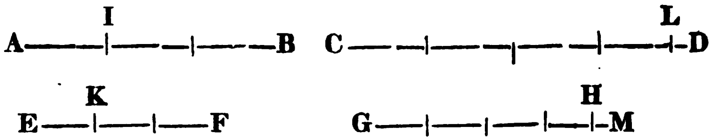
If the ratios of AB to CD, and of EF to GH, be not equal, let one of them, as of AB to CD, be the greater, and let CI a submultiple of CD be contained oftener in AB, than GK a like submultiple of GH is in EF (Hyp. and Def. 7. 5); and, since CD, GH are equimultiples of AB, EF (Hyp.), CI is contained oftener in CD, than GK is in GH (Cor. 1. 3. 5), which is absurd, CD, GH being equimultiples of CI, GK; therefore the ratio of AB to CD is not greater than that of EF to GH: in like manner it may be shewn, that the ratio of AB to CD is not less than that of EF to GH; it is therefore equal to it.

Theor. 2.—Magnitudes (AB, EF) have the same ratio, to equimultiples (CD, GH) of their equimultiples (AI, EM).



If the ratios of AB to CD, and of EF to GH, be not equal, let one of them, as of AB to CD, be the greater, and let CN a submultiple of CD, be contained oftener in AB, than GO a like submultiple of GH is in EF (Hyp. and Def. 7. 5); let CP, GQ be like submultiples of CN, GO, as AI, EM are of AB, EF, and CP, GQ are contained equally often in AI, EM, as CN, GO are in AB, EF (Cor. 2. 1. 5); but CN is contained oftener in AB, than GO is in EF, therefore CP is contained oftener in AI, than GQ is in EM, and therefore CP is contained oftener in CD, than GQ in GH (Cor. 1. 3. 5), which is absurd, CP, GQ being equisubmultiples of CN, GO, and therefore of their equimultiples CD, GH (3. 5); therefore the ratio of AB to CD is not greater than that of EF to GH: in like manner it may be shewn, that the ratio of AB to CD is not less than that of EF to GH; it is therefore equal to it.

Theor. 3.—If four magnitudes (AB, CD, EF, GH) be proportional, they are also proportional, taken inversely, [namely, CD is to AB, as GH to EF].



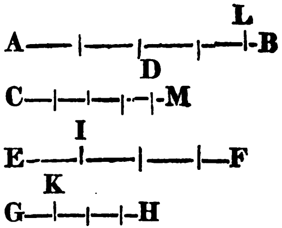
If the ratios of CD to AB, and of GH to EF, be not equal, let one of them, as of CD to AB, be the greater, and let AI a submultiple of AB, be contained oftener in CD, than EK a like submultiple of EF, is in GH (Hyp. and Def. 7. 5); let CL be the greatest multiple of AI, which is in CD, and GM a like multiple of EK, which is greater than GH; therefore the ratio of EF to GH is greater than that of EF to GM (8. 5), or, which is equal (Theor. 2. 15. 5), that of AB to CL, and therefore (7 and 8. 5), than that of AB to CD, contrary to the supposition and Cor. Def. 7. 5; therefore the ratio of CD to AB is not greater than that of GH to EF; and it may, in like manner, be proved, not to be less than the same; it is therefore equal to it.

Theor. 4.—If the first, of four magnitudes, have a greater ratio to the second, than the third has to the fourth; by inverting, the second has a less ratio to the first, than the fourth has to the third.

For, in the preceding theorem, from the supposition, of CD having a greater ratio to AB, than GH has to EF; it is proved, that the ratio of EF to GH is greater than that of AB to CD, or, which is the same, that the ratio of AB to CD is less than that of EF to GH.

PROP XVI. THEOR.

If four magnitudes of the same kind (AB, CD, EF, GH) be proportional; they are also proportional, taken alternately.

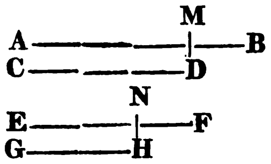


If the ratios of AB to EF, and of CD to GH be not equal, let one of them, as of AB to EF be the greater, and let EI a submultiple of EF be contained oftener in AB, than GK a like submultiple of GH is in CD (Def. 7. 5); let AL be the greatest multiple of EI which is in AB, and CM a like multiple of GK, which multiple is greater than CD; therefore the ratio of AL to CD is greater than that of AL to CM [8. 5], and therefore the ratio of AB to CD is greater than that of AL to CM [7, 8 and 13. 5 and Schol. 13. 5], and EI is to GK, as AL to CM [15. 5], therefore the ratio of AB to CD is greater than that of EI to GK [13. 5], or, which is equal [15. 5], than that of EF to GH, contrary to the supposition: therefore the ratio of AB to EF is not greater than that of CD to GH: in like manner, it may be proved, that the ratio of AB to EF is not less than that of CD to GH: since therefore the ratio of AB to EF is neither greater nor less than that of CD to GH, it is equal to it.

Cor. 1.—In this proposition, from supposing AB to have a greater ratio to EF, than CD has to GH, AB is proved to have a greater ratio to CD, than EF has to GH; and therefore, "if, of four magnitudes of the same kind, the ratio of "the first to the second, be greater than that of the third to the "fourth; by alternating, the ratio of the first to the third, is "greater than that of the second to the fourth."

PROP. XVII. THEOR.

If four magnitudes be proportional; the difference between the first and second, is to the second or first, as the difference between the third and fourth, is to the fourth or third, as the case may be.



Part 1.—Let AB, CD, EF, GH be four proportionals, the antecedents AB, EF being greater than the consequents CD, GH; the excess of AB above CD is to CD, as the excess of EF above GH is to GH.

On AB take AM equal to CD, and on EF, EN equal to GH (Post. 1. 5); and since any equisubmultiples whatever of CD, GH, or of their equals AM, EN are contained equally often in AB, EF (Hyp. and Def. 5. 5), they are contained equally often in the residues MB, NF (Cor. 6. 5), therefore MB, the excess of AB above CD, is to CD, as NF, the excess of EF above GH, is to GH (Def. 5. 5).

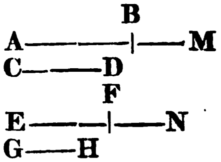
Part 2.—Let CD, AB, GH, EF be proportionals, the consequents AB, EF being greater than the antecedents CD, GH; a like construction being made, MB, the excess of AB above CD, is to AB, as NF, the excess of EF above GH, is to EF.

Because CD is to AB, as GH to EF (Hyp.), by inverting, AB is to CB, as EF to GH (Theor. 3. 15. 5), therefore by part 1, MB is to CD, as NF to GH, and, by inverting again, CD to MB, as GH to NF (Theor. 3. 15. 5), therefore any equisubmultiples whatever of MB, NF are contained equally often in CD, GH (Def. 5. 5), or in their equals AM, EN, and therefore in AB, EF (Cor. 2. 5), therefore AB is to MB, as EF to NF (Def. 5. 5), and inverting, MB to AB, as NF to EF (Theor. 3. 15. 5).

Part 3.—And since, by inverting, the second of four proportionals, has the same ratio to the first, as the fourth has to the third; by parts 1 and 2, the difference of the first and second is to the first, as the difference of the third and fourth is to the third.

PROP. XVIII. THEOR.

If four magnitudes be proportional; the compound of the first and second, is to the second or first, as the compound of the third and fourth, is to the fourth or third, as the case may be.



Part 1.—Let AB be to CD, as EF is to GH; the compound of AB and CD is to CD, as the compound of EF and GH is to GH.

To AB let BM be joined equal to CD, and to EF, FN equal to GH [Post. 1. 5]; and since any equisubmultiples whatever of CD, GH, or of their equals BM, FN, are contained equally often in AB, EF [Hyp. and Def. 5. 5], they are contained equally often in AM, EN [Cor. 2. 5], therefore AM, the compound of AB and CD, is to CD, as EN, the compound of EF and GH, is to GH [Def. 5. 5].

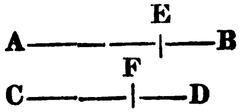
Part 2.—And since, by inverting, CD is to AB, as GH is to EF [Theor. 3. 15. 5], therefore, by part 1, the compound of AB and CD is to AB, as the compound of EF and GH is to EF.

Schol.—From this proposition, the preceding, and theor. 3. 15. 5, it follows, that, "if four magnitudes be proportional, "by converting, the first, is to the sum or difference of the first "and second, as the third, is to the sum or difference of the third "and fourth."

PROP. XIX. THEOR.

If the whole (AB), be to the whole (CD), as a part (AE) taken away, is to a part (CF) taken away; the residue (EB), is to the residue (FD), as the whole (AB), is to the whole (CD).

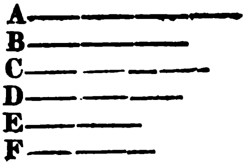
Because AB is to CD, as AE is to CF [Hyp.] by alternating, AB is to AE, as CD to CF [16. 5], and, by dividing, EB to AE, as FD to CF [17. 5], and, by again alternating, EB to FD, as AE to CF, or, [Hyp. and 11. 5], or, as AB to CD.



PROP. XX. THEOR. *(See Note).*

If there be three magnitudes (A, B, C), and other three (D, E, F), which, taken two and two in order, are in the same ratio; if the first (A), of the first magnitudes, be greater than the third (C), the first D), of the last magnitudes, is greater than the third (F); if equal, equal; and if less, less.

First, let A be greater than C; D is greater than F. For, since A is greater than C, the ratio of A to B, or which is equal [Hyp.], of D to E, is greater than that of C to B [8. 5]; and, since B is to C as E to F [Hyp.], by inverting, C is to B, as F is to E [Theor. 3. 15. 5]; therefore the ratio of D to E, having been shewn to be greater than that of C to B, is also greater than that of F to E, (13. 5), and therefore D is greater than F [10. 5].



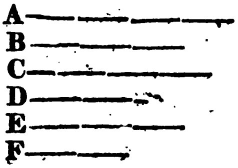
In like manner it may be shewn, that, if A be equal to C, D is equal to F; and if less, less.

PROP. XXI. THEOR.

If there be three magnitudes (A, B, C), and other three (D, E, F), which, taken two and two in a perturbate order, are in the same ratio; if the first (A), of the first magnitudes, be greater than the third (C), the first (D), of the last magnitudes, is greater than the third (F); if equal, equal; and if less, less.

First, let A be greater than C; D is greater than F.

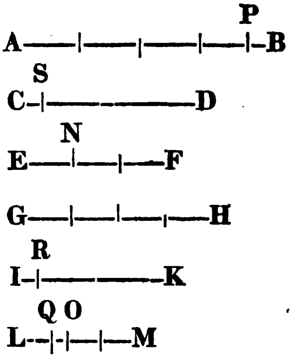
Because A is greater than C [Hyp.], the ratio of B to C is greater than of B to A [8. 5]; whence, D being to E, as B to C [Hyp.], the ratio of D to E is greater than of B to A; but, because A is to B, as E to F [Hyp.], by inverting, B is to A, as F is to E [Theor. 3. 15. 5]; whence, the ratio of D to E, having been shewn to be greater than that of B to A, is also greater than that of F to E (13. 5), therefore D is greater than F (10, 5).



In like manner it may be shewn, that, if A be equal to C, D is equal to F; and if less, less.

PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in order, are in the same ratio; by ordinate equality, the ratio of the first, of the first magnitudes, to the last, is the same, as the ratio of the first to the last, of the others.



First, let there be three magnitudes AB, CD, EF, and as many others GH, IK, LM; and let AB be to CD, as GH is to IK, and CD to EF, as IK to LM; AB is to EF, as GH to LM.

For, if the ratios of AB to EF and of GH to LM be not equal, let one of them if possible, as of AB to EF be the greater, and let EN a submultiple of EF be contained oftener in AB, the LO a like submultiqle of LM is in GH (Hyp. and Def. 7. 5); let AP be the greatest multiple of EN, which is in AB, and since a like multiple of LO, as AP is of EN, is greater than GH, a like submultiple of GH, as EN is of AP, is less than LO (Cor. 2. Ax. B. 5); let LQ be taken on LO equal to that submultiple, and let QO be the excess of LO above LQ; let IR be a submultiple of IK less than QO [Cor. Theor. at 7. 5], and CS a like submultiple of CD; and since CD is to EF, as IK is to LM [Hyp.], by inverting, EF is to CD, as LM is to IK [Theor. 3. at 15. 5], and therefore CS, IR being equisubmultiples of CD, IK, are contained equally often in EF, LM (Def. 5. 5), and therefore in their equisubmultiples EN, LO [Cor. 3. 3. 5]; whence, IR being less than QO, and therefore contained oftener in LO, than in LQ, the magnitude CS is contained oftener in EN, than IR in LQ; therefore AP, GH being equimultiples of EN, LQ, CS is contained oftener in AP, and therefore in AB, than IR is in GH Cor. 1. 3. 5); which is absurd [Hyp. and Def. 5. 5]; therefore the ratios of AB to EF, and of GH to LM, are neither of them greater than the other; they are therefore equal.

Let there be now four magnitudes A, B, C, D, and four others E, F, G, H; and let A be to B, as E to F; B to C, as F to G; and C to D, as G to H; A is to D, as E is to H.

A	B	C	D
E	F	G	H

Because A, B, C are three magnitudes, and E, F, G three others, which, taken two and two, are in the same ratio; by the preceding case, A is to C, as E is to G; whence, C being to D, as G is to H [Hyp.], by the same case, A is to D, as E is to H.

In like manner, the proposition might be demonstrated, if there were ever so many magnitudes.

Cor. 1.—In this proposition and by Def. 7. 5, from supposing AB to have a greater ratio to EF, than GH has to LM, and EF to be to CD, as LM to IK; AB is proved to have a greater ratio to CD, than GH has to IK; whence, it follows, that, "if, of two ranks of magnitudes, of three each, the ratio of "the first to the second be greater in one, than the other, and "the ratios of the second to the third, be equal in both; the "ratio of the first to the third, is greater in the former rank, "than in the latter."

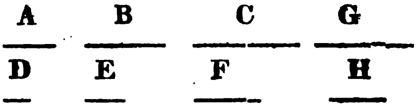
Cor. 2.—And the same thing being supposed, as in the preceding corollary, except that the ratio of EF to CD, instead of being equal to, be greater than, that of LM to IK; it might, in like manner, as in the demonstration of this 22d. proposition be shewn, that the ratio of AB to CD is greater than that of GH to IK; for CS would be at least as often contained in EF, as IR in LM (Hyp. and Cor. 4. 3. 5), and therefore, as often in EN, as IR in LO (Cor. 3. 3. 5); and therefore oftener than IR in LQ, and therefore oftener in AP or AB, than IR in GH (Cor. 1. 3. 5), and so the ratio of AB to CD is greater than that of GH to IK [Def. 7. 5]; therefore, "if, of two ranks of "magnitudes, of three each, the ratios of the first to the second, "and of the second to the third in one rank, be greater than the "corresponding ratios in the other, the ratio of the first to the "third in the former, is greater than that of the first to the third "in the other."

Cor. 3.—Ratios, duplicate, triplicate, &c. of equal ratios, are equal: This being a case of this proposition.

Cor. 4.—Ratios, duplicate, triplicate, &c. of unequal ratios, are unequal, those of the greater being the greater: this, in the case of duplicate ratios, being a case of the 2nd cor. above, and, in other cases, easily following from it.

Cor. 5.—Ratios, subduplicate, subtriplicate, &c. of equal ratios, are equal; for, if they were unequal, the ratios, which are duplicate, triplicate, &c. of them, would be unequal (by the preced. cor.), contrary to the supposition.

Cor. 6.—Ratios, sesquiplicate of equal ratios, are equal.

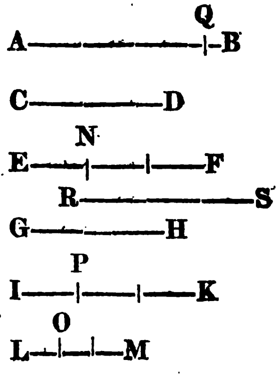


Let three magnitudes A, B, C be continually proportional, and three others D, E, F in the same ratio; let G be a mean proportional between B and C, and H between E and F; the ratios of A to G and of D to H, are sesquiplicate of the equal ratios of A to B and of D to E (Def. 14. 5); the ratios of A to G and of D to H are equal.

For the ratios of B to G and of E to H, being subduplicate of the equal ratios of B to C and E to F (Def. 14. 5), are equal (by the preced. cor.); whence, the ratios of A to B and of D to E being equal (Hyp.), A is to G, as D is to H (22. 5).

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in perturbate order, are in the same ratio; the ratio of the first, of the first magnitudes, to the last, is the same, as the ratio of the first to the last, of the others.

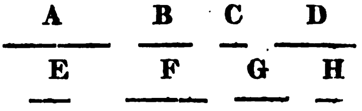


First, let there be three magnitudes AB, CD, EF, and as many others GH, IK, LM, and let AB be to CD, as IK is to LM, and CD to EF, as GH to IK; AB is to EF, as GH to LM.

For, if the ratios of AB to EF, and of GH to LM be not equal, let one of them, if possible, as of AB to EF be the greater, and let EN a submultiple of EF be contained oftener in AB, than LO a like submultiple of LM is in GH [Def. 7. 5]; let IP be a like submultiple of IK, as LO is of LM, AQ the greatest multiple of EN which is in AB, and RS a like multiple of IP.

The ratio of RS to IK is greater than that of GH to LM [Theor. 2. 15. 5 and 8. 5], therefore alternating, the ratio of RS to GH is greater than that of IK to LM [Cor. 16. 5], but IK is to LM, as AB is to CD (Hyp.), therefore the ratio of RS to GH is greater than that of AB to CD (13. 5); whence, GH being to IK, as CD is to EF (Hyp.), the ratio of RS to IK is greater than that of AB to EF (Cor. 1. 22. 5), which is absurd (Theor. 2. 15. 5 and 7 and 8. 5); therefore the ratios of AB to EF, and of GH to LM are not unequal, they are therefore equal.

Let there be now four magnitudes A, B, C, D, and four others E, F, G, H; and let A be to B, as G is to H; B to C, as F to G; and C to D, as E to F; A is to D, as E is to H.



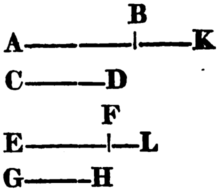
For, because there are three magnitudes A, B, C, and three others F, G, H, which, taken two and two in a perturbate order, are in the same ratio; A is to C, as F is to H (by the preced. part); and C is to D, as E is to F (Hyp.); therefore (by the same part) A is to D, as E is to H.

In like manner the proposition might be demonstrated, if there were ever so many magnitudes.

Schol.—By a like reasoning, as is used in the latter part of this proposition, it might be demonstrated, "if there be ever so "many magnitudes, and others equal to them in number, which "taken two and two, in any order whatever, are in the same "ratio; that the ratio of the first, of the first magnitudes, to "the last, is the same, as the ratio of the first to the last of the "others;" and therefore, that, "ratios compounded of equal "ratios (see Def. 13. 5), however disposed, are equal."

PROP. XXIV. THEOR.

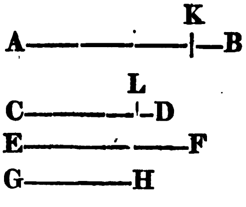
If to the antecedents (AB, EF) of four proportionals (AB, CD, EF, GH), magnitudes (BK, FL), which have the same ratio to their respective consequents (CD, GH), be added; the compounds (AK, EL), and consequents (CD, GH) are proportional.



Because BK is to CD, as FL is to GH (Hyp.), by inverting, CD is to BK, as GH to FL (Theor. 3. 15. 5); whence, AB being to CD, as EF to GH (Hyp.), by ordinate equality, AB is to BK, as EF to FL (22. 5); therefore, by compounding, AK is to BK, as EL to FL (18. 5), and therefore, BK being to CD, as FL to GH (Hyp.), by ordinate equality, AK is to CD, as EL to GH (22. 5).

PROP. XXV. THEOR.

If four magnitudes of the same kind (AB, CD, EF, GH) be proportional; the greatest (AB) and least (GH) together, are greater than the other two (CD, EF) together.



Take AK on AB equal to EF, and CL on CD equal to GH (Post. 1. 5); and, since AB is to CD, as EF is to GH (Hyp.), or, (Cor. 1. 7. 5 and 11. 5), as AK is to CL, AB is to CD, as KB is to LD (19. 5); and AB is greater than CD (Hyp.), therefore KB is greater than LD (Cor. 13. 5); and, because AK is equal to EF, and CL to GH, AK and GH together, are equal to CL and EF together (Ax. 2. 1); therefore, adding to them the unequals KB, LD, the magnitudes AB and GH together, are greater than CD and EF together (Ax. 4. 1).

BOOK VI.

DEFINITIONS.

1. *Similar rectilineal figures*, are such, as have all the angles of one, severally equal to those of the other, and the sides about the equal angles proportional.

2. A right line is said to be divided in *extreme and mean ratio*, when the whole is to the greater segment, as the greater segment to the less.

3. The *altitude* or *height* of any figure, is a perpendicular, let fall from the vertex or top on the base.

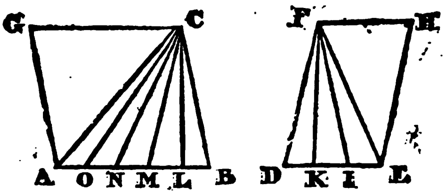
4. A parallelogram is said to be applied to a right line on which it is described.

5. A parallelogram, described on a part of any right line, is said to be applied to that right line, *deficient* by a parallelogram described on the residue of the right line, in the same angle, and of the same altitude.

6. And if a right line be produced, a parallelogram described on the compound of the right line and part produced, is said to be applied to the first mentioned right line, exceeding by the parallelogram described on the part produced, in the same angle, and of the same altitude.

PROPOSITION I. THEOREM.

Triangles (ABC, DEF), and parallelograms (BG, DH), which have the same altitude, are to each other, as their bases (AB, DE).



Part 1.—Divide DE into any number of equal parts DK, KI, IE (Cor. 7. 34. 1), and on AB take parts AO, ON, &c. as often as can be done, each equal to DK (3. 1), until a part LB remain less than DK, and join FK, FI, CO, CN, CM, CL.

Because the right lines DK, KI, IE, AO, ON, NM and ML are equal, the triangles FDK, FKI, FIE, CAO, CON, CNM, CML constituted on them, and being of the same altitude, are equal (38. 1), and the triangle CLB, being of the same altitude, and having its base LB less than DK, is less than the triangle FDK, therefore the triangle DKF is a like submultiple of DEF, as DK is of DE, and the triangle DKF and the right line DK, are contained equally often in the triangle ABC and right line AB.

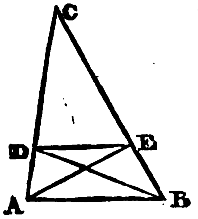
In like manner it may be proved, that any other equisubmultiples of the triangle DEF and right line DE, are contained equally often in the triangle ABC and right line AB; therefore the triangle ABC is to the triangle DEF, as AB is to DE [Def. 5. 5].

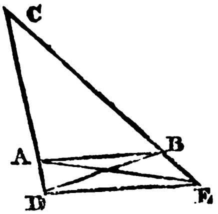
Part 2.—The parallelograms BG, DH, being double the triangles ABC, DEF (41. 1), are to each other, as these triangles (15. 5), and therefore, these triangles being to each other, as the bases AB, DE (by part 1), the parallelograms BG, DH are to each other, as the same bases (11. 5).

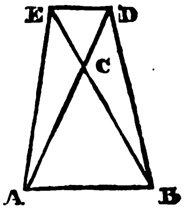
PROP. II. THEOR.

If a right line (DE, see all the fig. to this prop.) be drawn parallel to one of the sides (AB) of a triangle (ABC), it cuts the other sides (AC, BC), or these sides produced, proportionally.

And the right line (DE), which cuts two sides (AC, BC) of a triangle (ABC), or these sides produced, proportionally, is parallel to the remaining side (AB).







Part 1.—Let DE be parallel to AB; AD is to DC, as BE is to EC.

Join AE, DB, and because the triangles DAE, DBE are on the same base DE, and between the same parallels DE and AB, they are equal [37. 1]; therefore ADE is to CDE, as BDE is to CDE [7. 5]; but ADE is to CDE, as AD is to DC [1. 6], and BDE is to CDE, as BE is to EC (by the same); therefore AD is to DC, as BE is to EC.

Part 2.—Let AD be to DC, as BE to EC; DE is parallel to AB.

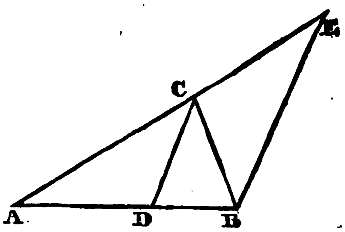
For, the same construction remaining, because AD is to DC, as BE to EC [Hyp.]; and AD is to DC, as the triangle ADE to the triangle CDE [1. 6]; and BE to EC, as the triangle BDE to the triangle CDE [by the same]; the triangle ADE is to CDE, as BDE to the same CDE [11. 5]; therefore the triangles ADE, BDE are equal [9. 5]; and they are on the same base DE, and to the same part; therefore DE is parallel to AB (39. 1).

Schol.—In the second part it is understood, that the homologous segments are similarly situated on the sides, or sides produced, of the triangle.

PROP. III. THEOR.

A right line (CD), bisecting an angle (ACB) of a triangle (ABC), cuts the opposite side into segments (AD, DB), having to each other the same ratio, as the other sides (AC, CB) of the triangle have to each other.

And a right line (CD), drawn from an angle (ACB) of a triangle (ABC), cutting the opposite side (AB) into segments (AD, DB), having to each other the same ratio, as the other sides (AC, CB) of the triangle have to each other, bisects that angle.



Part 1.—Through B, draw BE parallel to CD, meeting AC produced in E.

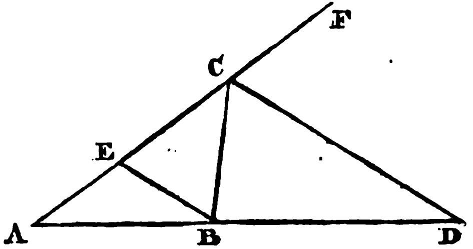
Because DC and BE are parallel, the angle CBE is equal to the alternate BCD (29. 1), or its equal (Hyp.) ACD, or, CE meeting the same parallels, to its equal, the internal remote angle CEB (29. 1]; therefore CB and CE are equal (6. 1); but, because DC is parallel to BE, AD is to DB, as AC is to CE (2. 6), whence, CB and CE being equal, AD is to DB, as AC is to CB.

Part 2.—The same construction remaining, AD is to DB, as AC to CB (Hyp.), and, because DC is parallel to BE, AD is to DB, as AC is to CE (2. 6), therefore AC is to CB, as AC is to CE (11. 5), of course CB and CE are equal (9. 5), and therefore the angle CBE is equal to the angle CEB (5. 1); whence, because of the parallels DC and BE, the angle ACD being equal to the internal remote CEB, and DCB to the alternate CBE (29. 1), the angles ACD and DCB are equal (Ax. 1. 1), and so the angle ACB is bisected by the right line CD.

Schol.—In the second part it is understood, that either segment of the divided side and the adjacent undivided side, are homologous terms.

Theorem.—If the external angle (BCF), of a triangle (ACB), made by producing one of its sides (AC), be bisected by a right line (CD) meeting the base produced; the segments (AD, BD) of the base produced, between its extremes (A, B), and the bisecting line, are to each other, as the other sides (AC, CB) of the triangle.

And if a right line (CD), be drawn from an angle (ACB) of a triangle (ABC), to a point (D) in the opposite side produced, the distances (AD, BD) of which point, from the extremes of that side, are to each other, as the other sides (AC, BC) of the triangle; the right so drawn bisects the external angle (BCF) formed at the first mentioned angle.



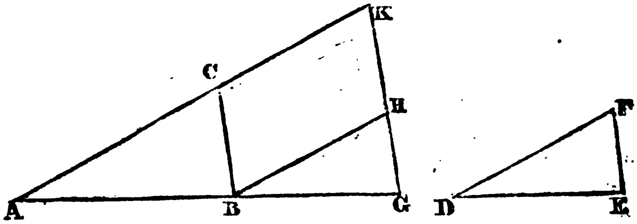
Part 1.—Through B draw BE parallel to CD.

Because BE and DC are parallel, the angle CBE is equal to the alternate angle BCD (29. 1); or, to its equal [Hyp.] DCF, or to the internal and opposite angle CEB [29. 1 and Ax. 1. 1], therefore CB is equal to CE [6. 1]; and, because in the triangle ACD, EB is parallel to CD, AE is to EC as AB is to BD [2. 6], and, by compounding, AC is to CE or its equal CB, as AD is to BD [18. 5].

Part 2.—The same construction remaining, AD is to BD, as AC to CB (Hyp.), and, because EB and CD are parallel, AD is to BD, as AC is to EC [2. 6], therefore AC is to CB, as AC to EC (11. 5), and so CB and CE are equal [9. 5], and the angle CBE is equal to the angle CEB [5. 1]; whence, because of the parallels EB and CD, the angle BCD being equal to the alternate CBE, and the angle FCD to the internal remote CEB (29. 1), the angles BCD and FCD are equal (Ax. 1. 1), and so the right line CD bisects the external angle BCF.

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles are proportional, the sides opposite equal angles, being homologous.



Let ABC and DEF be equiangular triangles, having the angle A equal to D, the angle ABC to E, and therefore [32. 1] the angle ACB to F. AC is to AB, as DF to DE, AB to BC, as DE to EF, and AC to CB, as DF to FE.

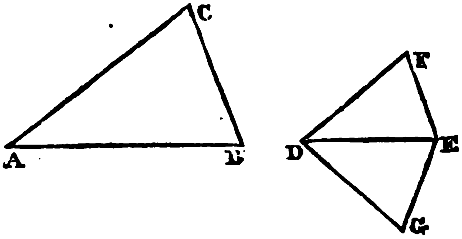
On AB produced take BG equal to DE, at the point B, with the right line BG, make the angle GBH equal to D or A, take BH equal to DF, and join GH: and since, in the triangles GBH, EDF, the sides GB, BH and the angle GBH, are severally equal to ED, DF and the angle EDF [Constr.], the triangles BGH and DEF are equal in all respects, and the angle G is equal to E [4. 1], or its equal [Hyp.] ABC; whence, the angles A and ABC being less than two right angles [17. 1], the angles A and G are less than two right angles, therefore AC and GH may be so produced as to meet [Theor. at 29. 1], let them be produced to meet, as in K; and, because the angle ABC is equal to G, BC and GK are parallel [28. 1], and, because the angle GBH is equal to A, BH and AK are parallel [by the same], therefore BHKC is a parallelogram, and of course, CK is equal to BH or DF, and HK to BC [34. 1].

And because, in the triangle AGK, BC is parallel to GK, AC is to CK, or its equal DF, as AB to BG or DE [2. 6], and by alternating, AC is to AB, as DF is to DE [16. 5]; and, because BH is parallel to AK, AB is to BG, or its equal DE, as KH, or its equal BC, is to HG or its equal EF [2. 6], and, by alternating, AB is to BC, as DE to EF [16. 5]; and since AC is to AB, as DF to DE, and AB to BC, as DE to EF, by ordinate equality, AC is to CB, as DF to FE [22. 5]: therefore the sides about the equal angles of the triangles ABC, DEF are proportional, the sides opposite equal angles being homologous [see def. 15. 5].

Schol.—Equiangular triangles are similar: see def. 1. 6.

PROP. V. THEOR.

If the sides of two triangles (ABC, DEF), about each of their angles, be proportional (AB to BC, as DE to EF; BC to AC, as EF to DF; and therefore, by ordinate equality, AB to AC, as DE to DF), the triangles are equiangular, having their equal angles opposite to the homologous sides.



At the extremes of any side DE, of either triangle, as DEF, make angles EDG and DEG equal to the angles A and B at the extremes of the side AB, which is homologous to DE; the remaining angle G of the triangle DEG, is equal to the remaining angle C of the triangle ABC (32. 1).

And, because the triangles ABC, DEG are equiangular, BA is to AC, as ED to DG (4. 6), and BA is to AC, as ED to DF (Hyp.), therefore ED is to DG, as ED to DF (11. 5), and so DG and DF are equal (9. 5); in like manner it may be proved, that EG and EF are equal, therefore the triangle DEG is equilateral to the triangle DEF, and of course equiangular to it [8. 1], and DEG is equiangular to ABC [Constr.], therefore the triangle ABC is equiangular to DEF, having the angle A equal to EDF, B to DEF, and C to F [Ax. 1. 1], namely, having those angles equal, which are opposite to the homologous sides.

PROP. VI. THEOR.

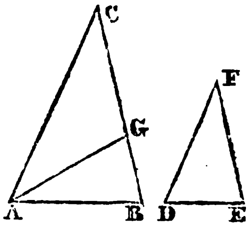
If two triangles (ABC, DEF, see fig. to prèced. prop.) have an angle (A) of one, equal to an angle (EDF) of the other, and the sides about the equal angles proportional (BA to AC, as ED to DF); the triangles are equiangular, having those angles equal, which are opposite to homologous sides.

With either leg DE, of either of the equal angles A and EDF, and at either extreme of it D, make the angle EDG equal to A, and at E, the angle DEG equal to B; the remaining angle G of the triangle DEG, is equal to the remaining angle C of the triangle ABC (32. 1).

And, because the triangles ABC, DEG are equiangular, AB is to AC, as ED to DG (4. 6), but AB is to AC, as DE to DF [Hyp.], therefore DE is to DG, as DE is to DF (11. 5), and so DG and DF are equal (9. 5); and the angles EDG and EDF, being each of them equal to A (Constr. and Hyp.], are equal to each other (Ax. 1. 1), and DE is common to the two triangles EDG, EDF, therefore the triangle EDG is equiangular to the triangle EDF (4. 1); and the triangle ABC is equiangular to the triangle EDG (Constr.); therefore the triangle ABC is equiangular to DEF, having the angle B equal to DEF, and E to F (Ax. 1. 1), and therefore having those angles equal, which are opposite to the homologous sides.

PROP. VII. THEOR.

If two triangles (ABC, DEF), have an angle (C) of one, equal to an angle (F) of the other, and the sides about two of the other angles proportional (BA to AC, as ED to DF), and the two remaining angles (B and E) either both less, or both not less than a right angle; the triangles are equiangular, having the angles equal, about which are the proportional sides.



Let first the angles B and E be both less than a right angle. The triangles ABC, DEF are equiangular, the angles CAB and FDE being equal.

For, if the angles CAB and FDE be not equal, let one of them, if possible, as CAB, be the greater, and at the point A, with the right line CA, make the angle CAG equal to D (23. 1).

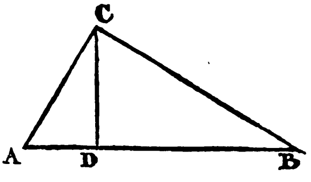
Because, in the triangles CAG, FDE, the angles C and F are equal [Hyp.], and the angles CAG and D also equal [Constr.], the remaining angles AGC and E are equal [32. 1]; therefore these triangles are equiangular, and of course CA is to AG, as FD to DE [4. 6]: and CA is to AB, as FD to DE [Hyp.], therefore CA is to AG, as CA to AB (11. 5), and so AG and AB are equal [9. 5]; therefore the angles AGB and ABG are equal [5. 1], and therefore both acute [Cor. 17. 1]; and because AGB is acute, AGC is obtuse [13. 1], and therefore the angle E, equal to AGC, is obtuse, which is absurd, the angle E being by supposition acute. Therefore the angles CAB and D are not unequal, they are therefore equal; and the angles C and F are equal [Hyp.], therefore the remaining angles B and E of the triangles ABC, DEF are equal (32. 1), which triangles are therefore equiangular, having the angles CAB and D, about which are the proportional sides, equal.

But if the angles B and E be not less than right angles, the same construction being made, it may in like manner be proved, that the angles B and AGB are equal; whence, the angle B being not less than a right angle (Hyp.) the angles B and AGB together are not less than two right angles; which is absurd (17. 1). Therefore, as in the former case, the angles CAB and D are not unequal, and are therefore equal, and the triangles ABC, DEF equiangular, having the angles CAB and D, about which are the proportional sides, equal.

Cor.—It follows from this proposition and 4. 1, that, "two "triangles, which have two sides of the one, severally equal to "two sides of the other, and the angles, opposite two of the "equal sides, equal, and the angles, opposite the other equal "sides, both less, or both not less, than a right angle, are "equal in all respects."

PROP. VIII. THEOR.

In a right angled triangle (ABC), a perpendicular (CD), let fall from the right angle (ACB) on the opposite side (AB), divides the triangle into parts (ADC, BDC), similar to the whole, and to each other.



For the triangles ACD, ABC, having the angle A common, and the angles ADC, ACB equal (Hyp. and Theor. 11. 1), are equiangular (32. 1); in like manner, the triangle BCD may be shewn to be equiangular to BAC; therefore the three triangles ACD, BCD and ACB are mutually equiangular (Ax. 1. 1), and therefore similar to each other (Cor. 4. 6).

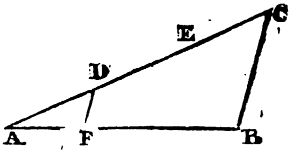
Cor. 1. Hence it is manifest, that, in a right angled triangle, a perpendicular (CD), let fall from the right angle, on the opposite side, is a mean proportional, between the segments (AD, DB) of the side on which it falls; and that the other sides (AC, CB) are mean proportionals between the adjacent segments (AD, DB), and the whole side (AB).

Cor. 2.—If a perpendicular (CD), let fall from an angle (ACB) of a triangle (ABC) on the opposite side (AB), be a mean proportional between the segments (AD, DB) of that side; the angle, from which the perpendicular is let fall, is a right one.

For since AD is to DC, as DC to DB, and the angles at D are right (Hyp.), the triangles ADC, CDB are equiangular (6. 6), having the angle ACD equal to B, and DCB to A; therefore the whole angle ACB is equal to the two angles A and B together, and therefore a right angle (32. 1).

PROP. IX. PROB.

From a given right line (AB), to cut off a required submultiple.

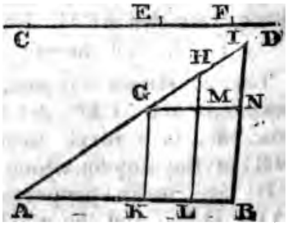


From either extreme A, of the given right line, draw AC, making any angle whatever with AB, take any point therein D, and, on AD produced, take AC a like multiple of AD, as AB is of the part to be cut off (3. 1), join BC, and draw DF parallel to BC, AF is the submultiple required.

For since FD is parallel to BC. BF is to FA, as CD to DA (2. 6), and by compounding, BA to FA, as CA to DA (18. 5), and, by inverting, AF to AB, as AD to AC (Theor. 3. 15. 5); but AD is the required submultiple of AC, therefore AF is the required submultiple of AB, for if AF were greater or less than such a submultiple, its ratio to AB would be greater or less than that of AD to AC (Theor. 1. 15. 5 and 8. 5), contrary to what has been demonstrated.

PROP. X. PROB.

To divide a given right line (AB) so, that the parts thereof, may have the same ratio to each other, as the parts (CE, EF, FD), of a given divided right line (CD).



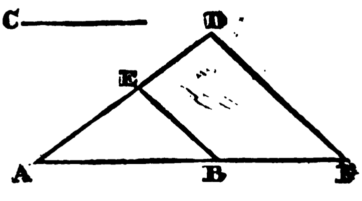
From either extreme A of the given right line, draw AI making any angle with AB, on which take AG equal to CE, GH to EF and HI to FD, draw IB, and parallel thereto GK, HL [31. 1], meeting AB in K and L; and through G draw GN parallel to AB meeting HL, IB in M and N; KM, LN are parallelograms, and therefore GM is equal to KL, and MN to LB, [34. 1], and of course KL is to LB, as GM to MN (Cor. 1. 7. 5), but, in the triangle GNI, because MH is parallel to NI, GM is to MN, as GH to HI [2. 6], therefore KL is to LB, as GH to HI (11. 5), or, (Cor. 1. 7. 5), as EF to FD; and, in the triangle ALH, because KG is parallel to LH, AK is to KL, as AG to GH, or, (Cor. 1. 7. 5), as CE to EF, and so AB is divided as required.

Cor. 1.—Hence it appears, how a given right line may be so divided, that its parts may have the same ratio to each other, as any given right lines have to each other, the construction being the same, only taking AG, GH and GI equal to these given right lines.

Cor. 2.—It follows also from the demonstration of this prop. that, if two or more right lines (KG, LH) be drawn parallel to one side (BI) of a triangle (ABI), all the segments of the other sides (AB, AI) are proportional; it being in this proposition demonstrated, from the parallelism of KG and LH to BI, that the segments of AB have the same ratio to each other, as those of AI, and the same reasoning being applicable to the case of more parallels.

PROP. XI. PROB.

To two given given right lines (AB and C) to find a third proportional.

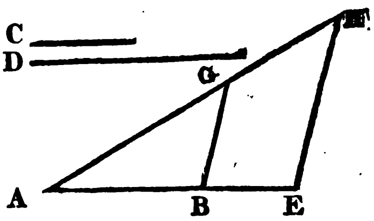


From an extreme A, of AB, draw AD, making any angle whatever with AB, take thereon AE equal to C, join BE, in AB produced take BF equal to C, and through F draw FD parallel to BE; ED is the third proportional required.

Because, in the triangle AFD, BE is parallel to FD, AB is to BF, as AE to ED (2. 6); but BF and AE are, each of them, equal to C (Constr.), therefore AB is to C, as C is to ED, and so ED is the third proportional required.

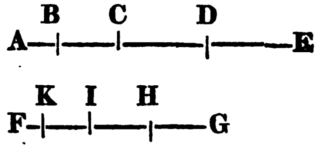
PROP XII. PROB.

To three given right lines (AB, C and D) to find a fourth proportional.



From either extreme A of the first right line AB, in any angle, draw the right line AH; on AB produced, take BE equal to C, and on AH, AG equal to D, join BG, and through E draw EH parallel to BG; GH is the fourth proportional required.

For, because, in the triangle AEH, BG is parallel to EH, AB is to BE, as AG is to GH (2. 6); but C is equal to BE, and D to AG (Constr.), therefore AB is to C, as D to GH.



Theorem 1.—A ratio of less inequality (as of AB to AC) may be so far continued, as to come to a magnitude greater than any given one.

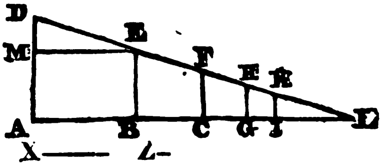
Let AB, AC, AD and AE be continually proportional; and, because AB is to AC, as AC to AD, by converting, AB is to BC, as AC to CD (Schol. 18. 5); and therefore because AC is greater than AB, CD is greater than BC (14. 5); in like manner it might be shewn, that DE is greater than CD; because therefore there is added to the first magnitude AB, magnitudes continually increasing, a magnitude would be at length come to, greater than any given one [Post. 2. 5].

Theor. 2.—A ratio of greater inequality (as of FG to FH, see fig. to preced. Theor.), may be so far continued, as to come to a magnitude less than any given one (AB).

Take AC a fourth proportional to FH, FG and AB [12. 6], and continue the ratio of AB to AC in the terms AD, AE [11. 6], till a term AE be come to greater than FG [by the preced. theor.], and let the ratio of FG to FH be continued through as many terms, and let the last term be FK.

Because then there are two ranks of proportional magnitudes equal in number, by ordinate equality, FG is to FK, as AE to AB [22. 5], but FG is less than AE [Constr.], therefore FK is less than AB (14. 5).

Problem.—Two right lines (AB and X), in a ratio of greater inequality, being given, to continue the ratio to any required number of terms, and to find the sum of the series continued through infinite terms.



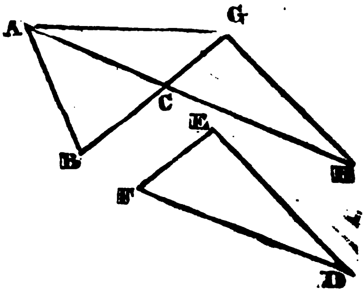
At the extremes of AB, draw AD and BE perpendicular to AB, take AD equal to AB, and BE and AM each equal to X, join EM, and because AM and BE are equal and parallel (Constr. and 28. 1), ME is parallel to AB (33. 1), and therefore perpendicular to AD (29. 1), and so the angle EDM less than a right angle (Cor. 17. 1), and the angles D and A together less than two right angles, therefore AB and DE may be so produced towards B and E, as to meet (Theor. 29. 1), let them be produced to meet, as in L; on AB produced take BC equal to X, and, having drawn CF perpendicular to AL, CF is a third proportional to AB and X; and, CG being taken equal to CF, the perpendicular GH is a fourth proportional, and so on; and AL is equal to the sum of the series, continued through infinite terms.

Part 1.—The triangles DAL, EBL, having the angles at A and B right, and the angle L common to both, are equiangular (32. 1), therefore AL is to AD, as BL to BE (4. 6); but AB is equal to AD, and BC to BE [Constr.], therefore AL is to AB, as BL to BC, and, by converting, AL is to BL, as BL to CL [Schol. 18. 5]; and, because AL is to AD, as BL to BE, by alternating, AL is to BL, as AD to BE [16. 5], in like manner it might be proved, that BL is to CL, as BE to CF; whence, AL having been proved to be to BL, as BL to CL; AD is to BE, as BE to CF; and, in like manner it may be shewn, that the other perpendiculars are proportional.

Part 2.—Because AD is to AL, as BE to BL, and AD is less than AL, BE is less than BL [Cor. 13. 5], and, in like manner, CF, GH, &c. may be shewn to be less than CL, GK, &c. therefore the whole series of proportionals is not greater than AL. And the whole series is not less than AL, for, if possible, let it be less than AL by any right line, as Z, and

PROP. XV. THEOR.

Of equal triangles (ABC, DEF), having an angle (ACB) of one, equal to an angle (F) of the other, the sides about the equal angles are reciprocally proportional (BC to FE, as FD to AC). And triangles (ABC, DEF), which have an angle (ACB) of one, equal to an angle (F) of the other, and the sides about the equal angles reciprocally proportional, are equal.



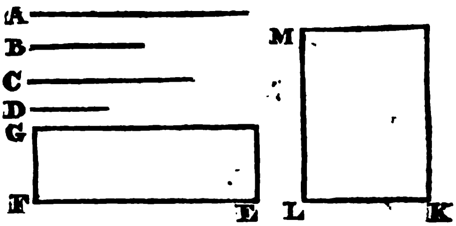
Part 1.—On BC and AC produced, take CG equal to FE, CH to FD, and join AG and GH.

The angle HCG is equal to BCA [15. 1], or, which is equal [Hyp.], the angle F; whence, the triangles HCG, DFE, having also the sides HC, CG severally equal to DF, FE (Constr.), are equal (4. 1); and the triangle ABC is equal to DFE (Hyp.), therefore the triangles ABC and HCG are equal (Ax. 1. 1); therefore the triangle ABC is to ACG, as HCG to the same ACG (7. 5); but the triangle ABC is to ACG, as BC to CG, and HCG to ACG, as HC to CA (1. 6), therefore BC is to CG, as HC to CA (11. 5), and FE is equal to CG, and FD to CH (Constr.), therefore BC is to FE, as FD to CA [7. 5].

Part 2.—The same construction remaining, the triangles HCG and DFE may, in like manner as in part 1, be proved equal; and, since BC is to FE, as FD is to CA [Hyp.], and CG is equal to FE, and CH to FD (Constr.), BC is to CG, as CH to CA (7. 5); but the triangle ABC is to ACG, as BC to CG, and HCG to ACG, as CH to CA (1. 6), therefore the triangle ABC is to ACG, as HCG to the same ACG (11. 5), and therefore the triangle ABC is equal to HCG (9. 5), or to its equal, as mentioned above, the triangle DFE.

PROP. XVI. THEOR.

If four right lines (A, B, C and D) be proportional, the rectangle under the extremes (A and D), is equal to the rectangle under the means (B and C). And four right lines (A, B, C and D), the rectangle under the extremes (A and D) of which, is equal to the rectangle under the means (B and C), are proportional.



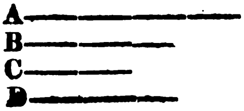
Part 1.—Draw the right lines FE and LK equal to A and B, draw LM and FG perpendicular to them, and equal to C and D, and complete the parallelograms GE and MK (Cor. 6. 34. 1).

Because then, in the parallelograms GE and MK, the angles F and L are equal, being right (Theor. at 11. 1), and FE to LK, as A to B (Constr. and Cor. 1. 7. 5), or, which is equal (Hyp.), as C to D, or, which is equal (Constr. and Cor. 1. 7. 5), as LM to FG, and therefore the sides about the equal angles F and L being reciprocally proportional, the parallelogram GE is equal to MK (14. 6).

Part 2.—The same construction remaining, because the parallelograms GE aad MK are equal (Hyp. and Cor. 3. 34. 1), the angles F and L being equal (Constr. and Theor. at 11. 1), FE is to LK, as LM to FG (14. 6), and therefore A is to B, as C is to D (Constr. Cor. 1. 7. 5 and 11. 5).

PROP. XVII. THEOR.

If three right lines (A, B and C) be proportional, the rectangle under the extremes (A and C), is equal to the square of the mean (B). And three right lines (A, B and C), the rectangle under the extremes of which is equal to the square of the mean, are proportional.

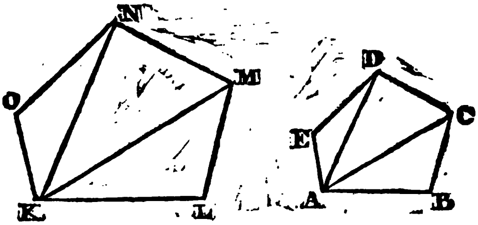


Part 1.—Take D equal to B (3. 1.); A is to B, as D is to C (Hyp. and 7. 5), therefore the rectangle under A and C is equal to the rectangle under B and D (16. 6), or, to the square of B.

Part 2.—The same construction remaining, the rectangle under A and C, being equal to the square of B (Hyp.), or, to the rectangle under B and D, A is to B, as D is to C (16. 6), and therefore, being B equal to D (Constr.), A is to B, as B is to C (7 and 11. 5).

PROP. XVIII. THEOR.

On a given right line (AB), to describe a rectilineal figure, similar and similarly posited to a given rectilineal figure (KLMNO).

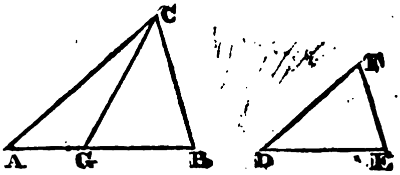


Join KM, KN, and, with the right line AB, at the points A and B, make the angles BAC and ABC equal to the angles LKM and KLM; the angles BAC and ABC, being equal to LKM and KLM [Constr.], are less than two right angles [17. 1], therefore AC and BC meet as in C [Theor. at 29. 1], and the remaining angle ACB of the triangle ABC is equal to KML [32. 1]; in like manner describe on AC, a triangle ADC equiangular to KNM; and so on.

The triangles ABC, ACD, &c. are equiangular to the triangles KLM, KMN, &c. (Constr.) and therefore similar (Cor. 4. 6); therefore the angle B is equal to L, and, adding equals to equals, the angle BCD to LMN (Ax. 2. 1), and, in like manner, the other angles of the figure ABCDE may be proved equal to the other angles of the figure KLMNO, therefore these figures are equiangular; and, because of the similar triangles ABC, KLM, AB is to BC, as KL to LM, and BC to CA, as LM to MK (Def. 1. 6), and, because of the similar triangles ACD, KMN, AC is to CD, as KM to MN, whence, BC having been shewn to be to CA, as LM to MK, by equality, BC is to CD, as LM to MN; and it may, in like manner, be proved, that the sides about the other angles of the figures ABCDE and KLMNO are proportional; whence, these figures, having been proved to be also equiangular, are similar (Def. 1. 6), and so there is described on AB, a rectilineal figure, similar and similarly posited to the rectilineal figure KLMNO, as was required.

PROP. XIX. THEOR.

Similar triangles (ABC, DEF), are to each other in a duplicate ratio of their homologous sides.

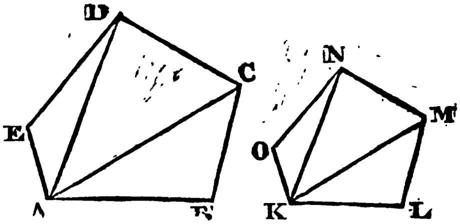


On any side of either of the triangles, as AB, produced if necessary, take AG a third proportional to AB, and the side DE, of the other triangle, which is homologous to it [11. 6], and join CG.

Because then AC is to AB, as DF to DE [Hyp. and Def. 1. 6], by alternating, AC is to DF, as AB to DE [16. 5], but AB is to DE, as DE to AG [Constr.], therefore AC is to DF, as DE to AG [11. 5], and the angles A and D are equal [Hyp. and Def. 1. 6], therefore the triangles CAG and FDE are equal [15. 6], and, of course, the triangle ABC has the same ratio to each of them [7. 5]; but the triangle ABC is to AGC, as AB to AG [1. 6], therefore the triangle ABC is to DEF, as AB to AG, or in a duplicate ratio of the homologous sides AB, DE [Constr. and Def. 14. 5].

PROP. XX. THEOR.

Similar polygons (ABCDE and KLMNO), may be divided into an equal number of similar triangles, having the same ratio to each other, as the polygons have; and the polygons are to each other in a duplicate ratio of their homologous sides.



Part 1.—Join AC, AD, KM, KN; and, because the polygons are similar, the angles B and L are equal, and AB is to BC, as KL to LM [Def. 1. 6], therefore the triangles ABC and KLM are similar [6. 6 and Cor. 4. 6]; and because the angles BCD, LMN are equal [Hyp. and Def. 1. 6], taking from them the equal angles BCA, LMK, the remaining angles ACD, KMN are equal; and because AC is to CB, as KM to ML, and BC to CD, as LM to MN [Hyp. and Def. 1. 6], by equality, AC is to CD, as KM to MN [22. 5]; and therefore, because of the equal angles between them, the triangles ACD, KMN are similar [6. 6 and Cor. 4. 6]; and in like manner the other triangles may be shewn to be similar.

Part 2.—Because the triangles ABC and KLM are similar, they are to each other in a duplicate ratio of AC to KM [19. 6], for a like reason, the triangles ACD and KMN are to each other in a duplicate ratio of AC to KM, therefore the triangle ABC is to KLM, as ACD to KMN [Cor. 3. 22. 5]; in like manner it may be proved, that the triangle ADE is to KNO, as ACD to KMN; therefore as one of the antecedents is to its consequent, so are all the antecedents to all the consequents [12. 6], or the polygon ABCDE to the polygon KLMNO.

Part 3.—Because the polygon ABCDE is to the polygon KLMNO, as the triangle ABC to the triangle KLM [by part 2], and the triangle ABC is to KLM in a duplicate ratio of AB to KL [19. 6], the polygon ABCDE is to the polygon KLMNO in a duplicate ratio of AB to KL [11. 5].

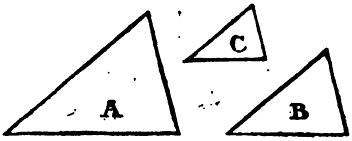
Cor. 1.—The homologous sides of similar rectilineal figures are to each other, in a subduplicate ratio of the figures themselves.

Cor. 2.—If three right lines be proportional, a rectilineal figure described on the first, is to a similar and similarly posited one on the second, as the first is to the third.

Cor. 3.—Similar rectilineal figures, are as the squares of their homologous sides, both the figures and squares being in a duplicate ratio of the same right lines.

PROP. XXI. THEOR.

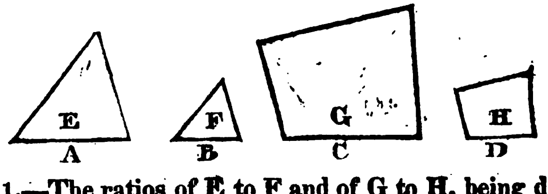
Rectilineal figures (A and B), which are similar to the same (C), are similar to each other.



Because the rectilineal figure A is similar to C [Hyp.], it is equiangular to it, and has the sides about the equal angles proportional [Def. 1. 6]; and because B is similar to C [Hyp.], it is equiangular to it, and has the sides about the eqnal angles proportional (Def. 1. 6); therefore A and B are equiangular to each other (Ax. 1. 1), and have the sides about the equal angles proportional (11. 5), and are therefore similar [Def. 1. 6].

PROP. XXII. THEOR.

If on the two first (A and B), of four proportional right lines (A, B, C and D), rectilineal figures (E and F), similar and similarly posited, be described, and also on the two last (as G and H on C and D); these figures are proportional. And if rectilineal figures so described on four right lines be proportional, the right lines themselves are proportional.

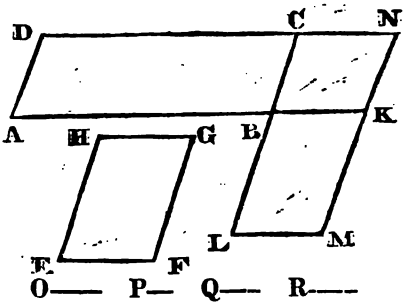


Part 1.—The ratios of E to F and of G to H, being duplicate of the equal ratios of A to B and of C to D [20. 6], are equal [Cor. 3. 22. 5].

Part 2.—The ratios of A to B and of C to D, being subduplicate of the equal ratios of E to F and of G to H [Cor. 1. 20. 6], are equal [Cor. 5. 22. 5].

PROP. XXIII. THEOR.

Equiangled parallelograms (AC and HF), are to each other, in a ratio, compounded of the ratios of the sides (of AB to HG, and of CB to HE).



On AB and CB produced, take BK equal to HG, and BL to HE, and complete the parallelograms BN and BM [Cor. 6. 34. 1].

In the parallelograms BM and HF, the angle LBK is equal to ABC [15. 1], or, which is equal [Hyp.] the angle H, and the sides about these angles are equal (Constr.], therefore these parallelograms are equal (Cor. 2 and 3. 34. 1); and the ratio of AC to BM, or its equal HF, is compounded of the ratios of AC to BN, and of BN to BM (Def. 13. 5), or, AB being to BK, as AC to BN (1. 6), and BC to BL, as BN to BM [by the same], in a ratio compounded of the ratios of AB to BK or HG, and of CB to BL or HE.

Cor. 1.—By a similar reasoning it may be proved, that triangles, which have an angle of one, equal to an angle of the other, are to each other, in a ratio, compounded of the ratios, of the sides including the equal angles.

Cor. 2.—A right line may be found, to which a given right line O has the same ratio, as two equiangled parallelograms AC and HF have to each other.

Take P a fourth proportional to AB, HG and O [12. 6], and Q a fourth proportional to CB, HE and P [by the same]. The right line O is to Q, in a ratio compounded of the ratios of O to P and of P to Q [Def. 13. 5], or of the ratios, equal to them [Constr.], of AB to HG and of CB to HE, and therefore [23. 6] as the parallelogram AC to HF. Therefore Q is the right line sought.

Cor. 3.—A square may be found, to which a given square has the same ratio, as two equiangled parallelograms AC and HF have to each other.

Let O be the side of the given square; find a right line Q, to which O has the same ratio, as AC has to HF [by preced. cor.]; find a mean proportional R, between O and Q [13. 6]. The square of O is to the square of R, as O is to Q]Cor. 2. 20. 6], or, which is equal (Constr.), as the parallelogram AC to HF. Therefore R is the side of the required square.

Cor. 4.—And since a rectangle may be made equal to any given rectilineal figure (45. 1), a right line or square may be found, to which a given right line or square has the same ratio, as any two given rectilineal figures have to each other.

Cor. 5.—The amount of the compound of two ratios is not altered, by shifting the consequents to different antecedents.

For the compound of the ratios of AB to HG and of CB to HE, is equal to the compound of the ratios of AB to HE and of CB to HG, each being equal to the ratio, which the parallelograms AC and HF have to each other (23. 6).

A—— B— C——— D——
E—— F——— G— H——

Cor. 6.—If, in two ranks, of four proportional right lines each (A to B, as C to D, and E to F, as G to H), any two corresponding extremes be equal; the other extremes are to each other, in a ratio, compounded of the ratios of the means to each other.

And if any two corresponding means be equal, the other means are to each other, in a ratio, compounded of the ratios of the extremes to each other.

Part 1.—Let the last extremes D and H be equal; A is to E, in a ratio, compounded of the ratios of B to F and of C to G.

The rectangle under A and D, is equal to that under B and C (16. 6), and the rectangle under E and H, to that under F and G (by the same); therefore the rectangle under B and C is to that under F and G, as the rectangle under A and D to that under E and H (Cor. 1. 7. 5), or, D and H being equal (Hyp.), as A is to E (1. 6 and 11. 5); but the rectangles under B and C, and under F and G, are to each other, in a ratio, compounded of the ratios of their sides (23. 6); therefore A is to E, in a ratio, compounded of the ratios of the same sides, namely, of B to F and of C to G.

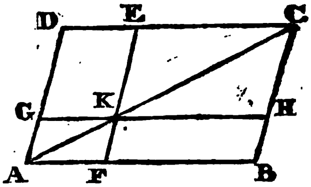
Part 2.—If the first extremes be equal; since, by inverting all the terms of each series, the first extremes become the last, and the contrary, the truth of the corollary is manifest from the preceding part.

Part 3.—And if two corresponding means be equal, since, by shifting the two first terms of each series, to the place of the two last, and the contrary, the extremes become means, and the means, extremes; the truth of the corollary, in this case, is manifest from the first and second parts.

Cor. 7.—Parallelograms or triangles are to each other, in a ratio, compounded of the ratios of their bases and heights; for they are equal to rectangles, or right angled triangles, of equal bases and heights.

PROP. XXIV. THEOR.

In every parallelogram (DB), parallelograms (EH, GF), which are about the diagonal, are similar to the whole, and to each other.



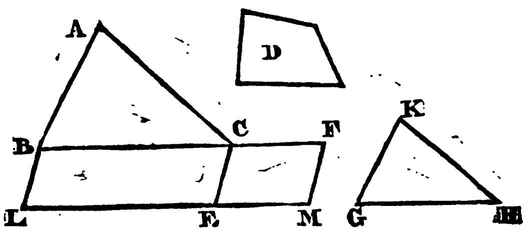
The parallelograms DB and GF, having the angle at A common, are equiangular (Cor. 2. 34. 1); and, because of the parallels GK and DC, the triangles AGK and ADC are equiangular (29. 1). and therefore similar (Cor. 4. 6), therefore AG is to GK, as AD to DC (Def. 1. 6); whence, the parallelograms GF and DB, having their other sides equal to AG, GK, AD and DC (34. 1), have the sides about the equal angles proportional, and, having been proved to be equiangular, are similar (Def. 1. 6).

In like manner it may be proved, that the parallelograms EH and DB are similar.

And, because the parallelograms GF and EH are similar to the same DB, they are similar to each other (21. 6).

PROP. XXV. THEOR.

To constitute a rectilineal figure, equal to a given one (D), and similar to another given one (ABC).



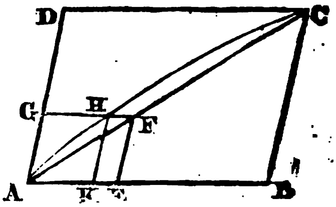
To any side BC, of the rectilineal figure ABC, apply the parallelogram BE, in any angle, equal to ABC (Cor. 45. 1), and to the right line CE, apply the parallelogram CM equal to D, having the angle FCE equal to CBL: BC and CF are in the same right line (29 and 14. 1); between BC and CF, find a mean proportional GH (13. 6); the figure described on GH, similar and similarly posited to ABC (18. 6), is equal to D.

For the parallelogram BE is to CM, as BC to CF (1. 6), or in a duplicate ratio of BC to GH (Constr. and Def. 14. 5), and, therefore, as the rectilineal figure ABC described on BC, to the similar and similarly posited figure GHK described on GH (20. 6): whence, the parallelogram BE being equal to ABC (Constr.), CM is equal to GHK (14. 5); but CM is equal to D (Constr.), therefore GHK is equal to D (Ax. 1. 1), and so there is constituted a figure GHK equal to D, and similar to ABC, as was required.

PROP. XXVI. THEOR. (*See Note*).

If two similar and similarly posited parallelograms, (DB, GE), have a common angle (DAB), they are about the same diagonal.

For if not, let the parallelogram DB, if possible, have its diagonal in a different right line from the diagonal AF of CE, as AHC, meeting one of the sides GF of the parallelogram GE in H, and through H draw HK parallel to AD.

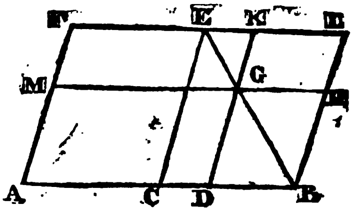


Because the parallelograms DB and GK, being about the same diagonal AHC, are similar (24. 6), GA is to AK, as DA is to AB (Def. 1. 6); but GA is to AE, as DA is to AB (Hyp. and Def. 1. 6); therefore GA is to AK, as GA is to AE (11. 5), and so AK and AE are equal [9. 5), part and whole, which is absurd. Therefore AHC is not the diagonal of DB. In like manner it may be shewn, that the diagonal AC, of the parallelogram DB, does not meet the sides GF or FE, of the parallelogram GE, in any other point, except F, therefore the parallelograms DB and GE are about the same diagonal AFC.

PROP. XXVII. THEOR.

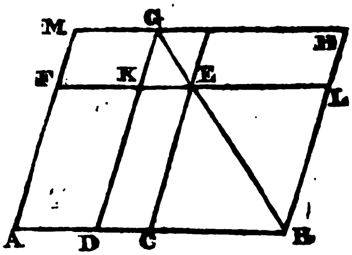
Of all parallelograms applied to the same right line (AC), deficient by parallelograms similar and similarly posited to that (AE or CL), which is described on the half line (AC or CB), that (AE), which is described on the half line, and is similar to its defect (CL), is the greatest.

First, let the parallelogram AG, applied to AB, and deficient by the parallelogram DH, similar to AE or CL, be described on a part AD greater than the half AC. The parallelogram AE is greater than AG.



Complete the parallelogram GL, and join EB; and, because

DH and CL are similar [Hyp.], EB is the diagonal of each (26. 6), and therefore CG is equal to GL (43. 1); adding to each DH, CH is equal to DL; but CM and CH are equal (Hyp. and 36. 1), therefore CM and DL are equal; add to each CG, and AG is equal to the gnomon CHK, and therefore less than CL, or its equal AE.



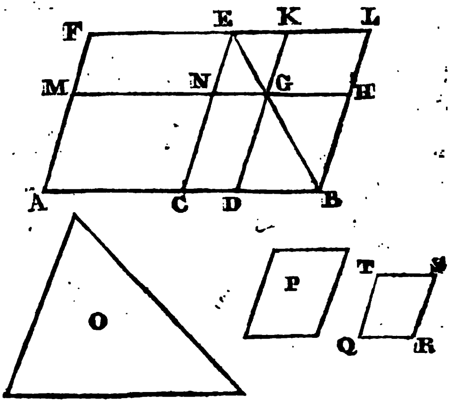
Secondly, let the parallelogram AG, applied to AB, and deficient by a parallelogram DH similar to AE or CL, be described on a part AD less than the half AC. The parallelogram AE is greater than AG.

Complete the parallelogram EH, and join GB, which, because DH and CL are similar, is the diagonal of each (26. 6), and therefore DE is equal to EH (43. 1); but FE and EL are equal (Hyp. and 34. 1), therefore the parallelograms ME, EH are equal (36. 1), and of course DE is equal to ME (Ax. 1. 1), and therefore greater than its part MK; adding to each AK, the whole AE is greater than the whole AG (Ax. 4. 1).

Schol.—It is manifest, that the parallelogram EG is, in both cases, the excess of AE above AG, which of course is diminished by diminishing CD.

PROP. XXVIII. THEOR.

To a given right line (AB), to apply a parallelogram, equal to a given rectilineal figure (O), and deficient by a parallelogram similar to a given parallelogram (P). But the rectilineal figure ought not to be greater than the parallelogram applied to half the given right line, whose defect is similar to the given parallelogram.



Bisect AB in C, on AC, describe the parallelogram AE, similar and similarly posited to the given one P (18. 6), and complete the parallelogram AL (Cor. 6. 34. 1).

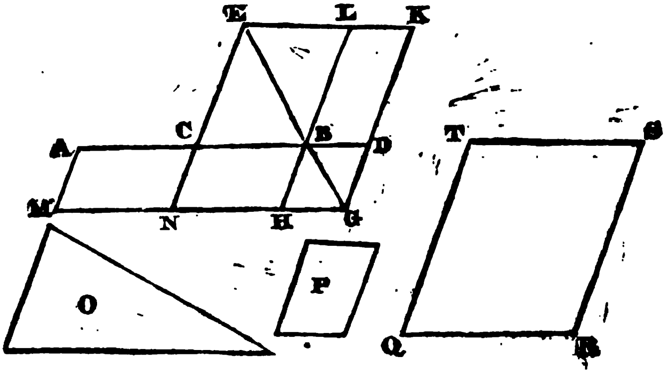
The parallelogram AE, is either equal to, or greater than, the rectilineal figure O [Hyp.

If equal, what was required is done. For to the right line AB, the parallelogram AE is applied, equal to O, and deficient by CL, similar to P.

If AE be greater than O, make the parallelogram QRST equal to the excess of AE above O, and similar and similarly posited to P (14. 2. Cor. 2. 47. 1 and 25. 6); and, since the parallelogram QS is less than AE, it is less than CL, which is equal to AE (36. 1); but QS is similar to CL, being each of them similar to P (21. 6); therefore the sides QT, TS are less than the sides homologous to them CE, EL; take from CE, EL, the parts EN equal to QT and EK to TS, and complete the parallelogram NK (Cor. 6. 34. 1), which is similar to CL, the parallelograms NK, CL being each of them similar to QS, and therefore to each other (21. 6), and, being similarly posited, are about the same diagonal (26. 6); let this diagonal be EGB, and produce KG to D, and NG to M and H; and, because the parallelogram CL is equal to O and QS together (Constr.), and NK is equal to QS, the gnomon CHK is equal to O; but CG and GL are equal [43. 1], and, adding to each DH, the parallelogram CH is equal to DL; and AN is equal to CH (Constr. and 36. 1), therefore AN is equal to DL, adding to each CG, AG is equal to the gnomon CHK; but it has been proved, that the gnomon CHK is equal to O, therefore AG is equal to O, and DH is similar to CL (24. 6), and therefore to P (Constr. and 21. 6). Therefore there is applied to the given right line AB, a parallelogram AG equal to O, and deficient by a parallelogram DH, similar to P, as was required.

PROP. XXIX. PROB.

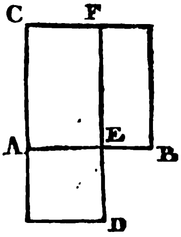
To a given right line (AB), to apply a parallelogram, equal to a given rectilineal figure (O), and exceeding by a parallelogram similar to a given parallelogram (P).



Bisect AB in C, and on CB make the parallelogram CL similar and similarly posited to P [18. 6]; make the parallelogram QS equal to CL and O together, and similar and similarly posited to P (14. 2. Cor. 1. 47. 1 and 25. 6); and because QS is greater than CL, its sides QT and TS are greater than the homologous sides CE and EL of the parallelogram CL; on these produced, take EK equal to TS and EN equal to TQ; complete the parallelogram NK (Cor. 6. 34. 1), which is equal and similar to QS, and therefore similar to CL; but it is also similarly posited to it, therefore NK and CL are about the same diagonal (26. 6); draw their diagonal EBG, through A, draw AM parallel to CN, meeting GN produced in M, and let LB, CB produced, meet MG, GK in H and D; and, because QS is equal to CL and O together [Constr.], and NK equal to QS, [Cor. 3. 34. 1), NK is equal to CL and O together; take away the common part CL, and the gnomon NDL is equal to O; but, because of the equals AC, CB, the parallelogram MC is equal to NB (36. 1 , or its equal (43. 1) BK; adding to each ND, the parallelogram MD is equal to the gnomon NDL, or its equal O; and HD is similar to CD (24. 6), and therefore (Constr.) to P. There is therefore applied to the right line AB, a parallelogram MD, equal to O, and exceeding by a parallelogram HD similar to P, as was required.

PROP. XXX. PROB.

To cut a given right line (AB) in extreme and mean ratio.



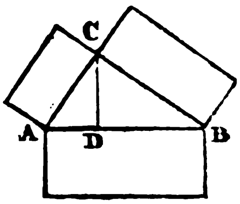
On AB describe the square CB (46. 1), and to AC apply the parallelogram CD equal to BC, exceeding by a figure AD similar to BC (29. 6); and because AD is similar to the square BC, it is itself a square; and since BC is equal to CD, taking from each the common part CE, BF is equal to AD; but it is also equiangular to it, therefore EF is to ED, as AE is to EB [14. 6]; but EF and ED are equal to AB and AE, therefore AB is to AE, as AE is to EB, and so AB is cut in extreme and mean ratio (Def. 2. 6).

Otherwise.

Divide AB in E, so that the rectangle under AB and EB may be equal to the square of AE (11. 2), and AB is to AE, as AE to EB (17. 6), and so AB is cut in extreme and mean ratio (Def. 2. 6).

PROP. XXXI. THEOR.

If, on the sides of a right angled triangle (ABC), similar and similarly posited rectilineal figures be described; that which is described on the side (AB) opposite the right angle, is equal to the other two taken together.



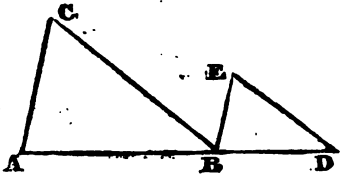
Let fall the perpendicular CD from the right angle on the opposite side AB, and AB is to AC, as AC to AD (Cor. 1. 8. 6); therefore the figure on AB is to the similar one on AC, as AB to AD [Cor. 2. 20. 6]; in like manner it may be proved, that the figure on AB is to that on BC, as AB to BD; therefore the figure on AB is to the two figures on AC and CB together, as AB to AD and DB together (Theor. 3. 15. 5 and 24. 5); but AB is equal to AD and DB together, therefore the figure described on AB is equal to the similar figures on AC and CB together (Cor. 13. 5).

Cor.—Hence any number of similar rectilineal figures being given, a rectilineal figure may be found equal to them all, by the aid of Cor. 1. 47. 1. And two similar rectilineal figures being given, one may be found equal to their difference, by the aid of Cor. 2. 47. 1.

PROP. XXXII. THEOR.

If two triangles (ABC, BDE), which have two sides (AC, CB) of one, proportional to two sides (BE, ED) of the other (AC to CB, as BE to ED), be so joined at an angle, that the homologous sides may be parallel (AC to BE, and CB to ED), and the sides (CB, BE), which are not homologous, may constitute the angle at which they are joined, the remaining sides (AB and BD) are in the same right line.

For, because AC and BE are parallel, the alternate angles C and CBE are equal (29. 1); and, in like manner, because CB and ED are parallel, the angles E and CBE are equal; therefore the angle C is equal to E; and

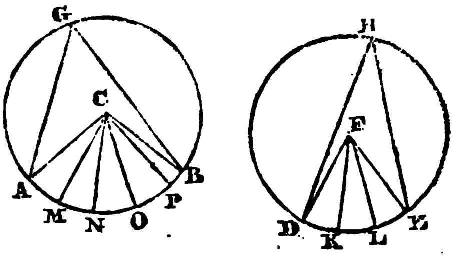


because the sides of the triangles ACB, BED about these equal angles are proportional [Hyp.], these triangles are equiangular (6. 6), and therefore the angle A is equal to EBD; and the angle C is equal to CBE, therefore the angles A and C together, are equal to the angle CBD; to each add the common angle ABC, and the three angles A, C and ABC are equal to the two angles CBD and CBA; but the three angles A, C and ABC of the triangle ABC are equal to two right angles (32. 1), therefore the angles CBD and CBA are together equal to two right angles, and of course AB and BD are in the same right line (14. 1).

PROP. XXXIII. THEOR.

In equal circles, angles, whether at the centre (as ACB, DFE), or at the circumference (as AGB, DHE), are to each other, as the arches (AB, DE), on which they stand. As are also the sectors (ACB, DFE).

Part 1.—Let DK, KL, LE be any number of equal parts, into which the arch DE is divided, on AB take as many parts AM, MN, NO, OP, as are in AB equal to DK (Cor. 1. 4), until a part PB remain less than DK, and join FK, FL, CM, CN, CO, CP.



Because the arches DK, KL, LE, AM, MN, NO and OP are equal, the angles DFK, KFL, LFE, ACM, MCN, NCO and OCP are equal (27. 3), and the arch PB being less than DK, the angle PCB is less than DFK, therefore the angle DFK and arch DK are equisubmultiples of the angle DFE and arch DE, and are contained equally often in the angle ACB and arch AB. In like manner it may be proved, that any other equisubmultiples of the angle DFE and arch DE, are contained equally often in the angle ACB and arch AB; therefore the angle ACB is to the angle DFE, as the arch AB is to the arch DE (Def. 5. 5).

Part 2.—And the angles AGB, DHE being halves of the angles ACB, DFE [20. 3], are to each other, as these angles [15. 5], and therefore (by part 1 and 11. 5), as the arches AB, DE.

Part 3.—The sectors ACB, DFE may be proved to be to each other, as the arches AB, DE, in the same manner, as in part 1, the angles ACB, DFE are proved to be to each other in that ratio; only substituting for the words, *angle* and *angles*, the words *sector* and *sectors*, and for 27. 3, Cor. 1. 29. 3.

Cor. 1.—An angle (ACB) at the centre of a circle, is to four right angles, as the arch (AB) on which it stands, is to the whole circumference.

For the angle ACB is to a right angle, as the arch, on which it stands, is to a fourth part of the circumference (33. 6); and therefore (Theor. 1. 15. 5 and 22. 5), the angle ACB is to four right angles, as the arch AB is to the whole circumference.

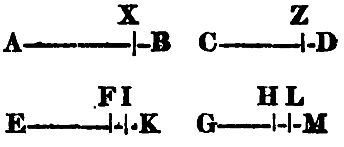
Cor. 2.—Arches of unequal circles, which subtend equal angles at the centre, or at the circumference, have equal ratios to their whole circumferences.

Part 1.—If the equal angles be at the centre, the ratios of the arches to their whole circumferences, are each of them equal to that of either of the equal angles to four right angles [by preced. Cor.], and therefore to each other [11. 5].

Part 2.—If the equal angles be at the circumference, since angles at the centre, insisting on the same arches, are double to them (20. 3), and therefore equal (Ax. 6. 1), the ratios of the arches to their whole circumferences are equal by part 1.

Theor. 1.—(*See note.*) If there be two magnitudes (AB and CD), and two others (EF and GH) both severally less than them, and the latter, by repeated augmentations, always, in such augmentations, retaining the same ratio to each other, and being less than the former, approach continually nearer and nearer to equality with the same former, so as at length to be deficient of them, by magnitudes less than any given ones; the former are to each other in the same ratio.

Let EF, EI, EK, &c. and GH, GL, GM, &c. be the continually increasing magnitudes; and if the ratio of AB to CD be not equal to that of EF to GH, let it, if possible, be greater or less than it; and first, let it be greater, and let AX be a magnitude, which is to CD, as EF to GH, which is

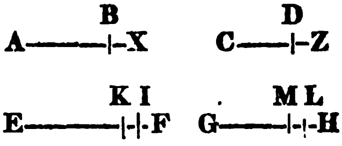


less than AB (Hyp. and 10. 5), let XB be the excess of AB above AX, and let EF, EI, EK, GH, GL, GM, &c. be continued, till EK may want of AB by a magnitude less than XB (Hyp.), therefore EK is greater than AX, and the ratio of EK to CD is greater than that of AX to CD [8. 5], or [Constr.] of EF to GH, or, which is equal (Hyp.), of EK to GM; since therefore the ratio of EK to CD is greater than that of the same EK to GM, CD is less than GM (10. 5), contrary to the supposition. Therefore the ratio of AB to CD is not greater than that of EF to GH.

Let now the ratio of AB to CD be, if possible, less than of EF to GH, and let CZ be a fourth proportional to EF, GH and AB, the magnitude CZ is less than CD [Hyp. and 10. 5], let ZD be the excess of CD above CZ, and let EF, EI, GH, GL, &c. be continued, till GM may want of CD by a magnitude less than ZD (Hyp.), therefore GM is greater than CZ, and the ratio of EK to CZ greater than of EK to GM (8. 5), or (Hyp.) of EF to GH, or (Constr.), of AB to CZ; since therefore the ratio of EK to CZ is greater than of AB to the same CZ, the magnitude EK is greater than AB [10. 5], contrary to the supposition; therefore the ratio of AB to CD is not less than of EF to GH; and it has been shewn, not to be greater than it; therefore the ratio of AB to CD, is equal to that of EF to GH.

Theor. 2.—If there be two magnitudes (AB and CD), and two others (EF and GH) both severally greater than them, and the latter, by repeated diminutions, always, in such diminutions, retaining the same ratio to each other, and remaining greater than the former, approach nearer and nearer to equality with the same former, so as at length to exceed them, by magnitudes less than any given ones; the former are to each other in the same ratio.

Let EF, EI, EK, &c. and GH, GL, GM, &c. be the continually decreasing magnitudes; and, if the ratio of AB to CD be not equal to that of EF to GH, let it, if possible, be greater or less than it; and first, let it be greater, and let CZ be a fourth proportional to EF, GH and AB, the magnitude CZ is greater than CD (Hyp. and 10. 5), let DZ be the excess of CZ above CD, and let EF EI, GH, GL, &c. be continued, till GM exceed CD by a magnitude less than



DZ [Hyp.], therefore GM is less than CZ, and the ratio of EK to CZ is less than of EK to GM [8. 5], or [Hyp.], of EF to GH, or [Constr.] of AB to CZ; since then the ratio of AB to CZ is greater than of EK to the same CZ, the magnitude AB is greater than EK (10. 5), contrary to the supposition, therefore the ratio of AB to CD is not greater than of EF to GH.

Let now the ratio of AB to CD be, if possible, less than of EF to GH, and let AX be to CD, as EF to GH, AX is greater than AB (Hyp. and 10. 5), let BX be the excess of AX above AB, and let EF, EI, FH, FL, &c. be continued, till EK exceed AB by a magnitude less than BX [Hyp.], therefore AX is greater than EK, and the ratio of GM to AX is less than of GM to EK (8. 5), or (Hyp.) of GH to EF, or (Constr. and Theor. 3. 15. 5) of CD to AX; since therefore the ratio of CD to AX is greater than of GM to the same AX, the magnitude CD is greater than GM (10. 5), contrary to the supposition; therefore the ratio of AB to CD is not less than of EF to GH, and it has been shewn, not to be greater than it; therefore the ratio of AB to CD, is equal to that of EF to GH.

ELEMENTS OF PLAIN TRIGONOMETRY.

TRIGONOMETRY, is that science, whereby, from having certain sides or angles of triangles given, the other sides and angles may be found.

As these triangles are of two kinds, namely, those which, being in a plain, are bounded by right lines, and those which, being on the surface of the globe, are bounded by arches of great circles of the globe; trigonometry is usually divided into two kinds, Plain and Spherical.

Plain Trigonometry, is that, which treats of plain triangles.

Note, as there will be occasion hereafter of making other citations, besides those from the preceding elements of Euclid, citations from these will be marked *Eu,* from Plain Trigonometry, *Pl. Tr.*

DEFINITIONS.

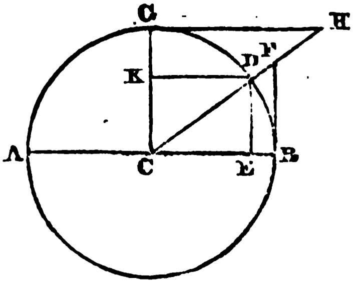
1. A *degree* of a circle, is an arch thereof, equal to a three hundred and sixtieth part of its whole circumference; a *minute,* an arch, equal to the sixtieth part of a degree; a *second,* an arch, equal to the sixtieth part of a minute, and so forth.

2. An arch of a circle, described from the concourse of the legs of a rectilineal angle, as a centre, included between these legs, is said to be, the *measure* of the angle; which is said to be, of as many degrees, minutes, &c. as is the arch so included.

Schol.—The number of degrees, minutes, &c. included between the legs of the angle, is not varied, by varying the length of the radius; since the intercepted arches whether small or great, have the same ratio to their whole circumferences, by Cor. 2. 33. 6. *Eu.* and therefore contain an equal number of these degrees, minutes, &c. (By converse of Theor. 2. 15. 5. *Eu*).

3. A *quadrant* of a circle, is a fourth part of its circumference.

4. The difference, of an angle from a right angle, or of an arch from a quadrant, is called the *complement*, of that angle or arch. Thus, if CG be perpendicular to AB, the angle DCG is the complement of the angle BCD or ACD; and the arch GD, of the arch BD or AGD.



5. The complement, of an angle to two right angles, or of an arch to a semicircle, is called the *supplement*, of that angle or arch. Thus the angle ACD is the supplement of BCD, and the arch AGD of BD.

6. The *sine* of an arch (BD or AGD), or of the angle (BCD or ACD) of which it is the measure, is a perpendicular (DE), let fall from one extremity (D) of the arch, on the diameter (AB), passing through its other extremity (B or A).

7. The *versed sine* of any arch (BD), or of its corresponding angle (BCD), is the segment (EB) of the diameter (AB) drawn through one extremity (B) of the arch, between that extremity and the perpendicular (DE) let fall on the diameter from the other extremity [D].

8. The *tangent* of an arch (BD or AGD), or of its corresponding angle (BCD or ACD), is a right line (BF), drawn from one extremity (B) of the arch, and touching it in that extremity, to the diameter (CDF) produced, which passes through its other extremity (D).

9. And the segment (CF) of the diameter, so produced and meeting the tangent, between the centre and tangent, is called the *secant* of the same arch or angle.

10. The cosine of any arch or angle, is the sine of its complement to a quadrant or right angle; and the *cotangent* of any arch or angle, is the tangent, and the *cosecant* of any arch or angle, the secant of such complement.

Thus KD or CE is the cosine, GH the cotangent, and CH the cosecant, of the arch BD or angle BCD; being the sine, tangent and secant, of the arch GD or angle GCD.

Cor. 1 *to these definitions.*—The sine, tangent or secant, of any arch or angle, is the sine, tangent or secant of its supplement, or complement to a semicircle or two right angles.

For it is manifest from these definitions, that the same right lines are the sine, tangent and secant of the arches BD and AGD, and of the angles BCD and ACD.

Cor. 2.—The sine of a quadrant or right angle, is equal to the radius.

Cor. 3.—The tangent of half a quadrant or half a right angle is equal to the radius; for if the angle BCF were half a right angle, the angle CBF being right (18. 3 *Eu.*), the angle CFB would be half a right angle (32. 1 *Eu.*), whence, the angles BCF, BFC being equal, the tangent BF would be equal to CB (6. 1 *Eu.*), or to radius.

Cor. 4.—The radius is a mean proportional between the tangent (BF) and cotangent (GH) of any arch (BD).

The triangles CBF, HGC, having the angles at B and G right, and the angles at C and H equal, being alternate angles formed by CH meeting the parallels CB, GH (29. 1 *Eu.*), are equiangular (32. 1 *Eu.*), therefore BF is to CB, as CG or CB to GH (4. 6 *Eu*).

Cor. 5.—The tangents of any two arches of the same circle, and, of course, of any two angles, are reciprocally as their cotangents.

For the rectangle under the tangent and cotangent of the arch BD, is equal to the rectangle under the tangent and cotangent of any other arch of the circle, each of these rectangles being equal to the square of radius [preced. cor. and 17. 6 *Eu.*], and therefore the tangent of BD is to the tangent of that other arch, as the cotangent of the same arch is to the cotangent of BD (16. 6 *Eu*).

Cor. 6.—The radius is a mean proportional, between the cosine (CE) and secant (CF), or between the sine (DE) and cosecant (CH), of any arch (BD).

Since ED and BF are parallel, CE is to CB, as CD or CB is to CF (2. 6 and Schol. 18. 5 *Eu*).

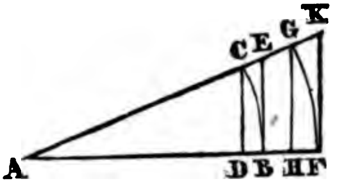
And since KD is parallel to GH, CK or DE is to CG, as CD or CG is to CH. [2. 6. *and Schol.* 18. 5. *Eu*].

Cor. 7.—The sines of any two arches of the same circle, are reciprocally as their cosecants; and the cosines, reciprocally as the secants.

Since the rectangle under the sine and cosecant of BD, is equal to the rectangle under the sine and cosecant of any other arch of the circle, each of these rectangles being equal to the square of radius [preced. cor. and 17. 6 *Eu.*], the sine of BD is to the sine of that other arch, as the cosecant of the same arch is to the cosecant of BD [16. 6 *Eu*].

In like manner it may be shewn, from cor. 6 above, and 16 & 17. 6 *Eu.* that the cosines of any two arches, are reciprocally as their secants.

Cor. 8.—The ratio of the radius, to the sine tangent or secant of any angle (as A), is the same, whatever be the dimension or magnitude of the radius.



On either leg, as AB, of the angle A, from the point A, take any unequal parts as AB, AF, and from the centre A, at the distances AB, AF, let arches of circles be described meeting the other leg of the angle A, in C and G; from the points C, G, let fall the perpendiculars CD, GH on AF; and at the points B, F, raise the perpendiculars BE, FK meeting AC produced in E and K; CD and GH are the sines of the arches BC, FG (Def. 6. *Pl. Tr.*), BE and FK the tangents [Def. 8. *Pl. Tr.*], and AE and AK the secants of the same arches (Def. 9. *Pl. Tr.*); which arches are the measures of the angle CAB. The radius has in both cases the same ratio to the corresponding sine, tangent or secant.

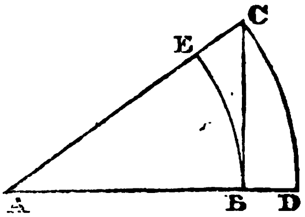
The triangles ADC, AHG are, because of the right angles at D and H, and the common angle at A, equiangular (32. 1 *Eu.*), therefore AC is to CD, as AG is to GH [4. 6 *Eu.*]; but AC, AG are the radiuses, and CD, GH the sines of the arches CB, GF [Def. 6. *Pl. Tr.*], which are measures of the angle A.

And the triangles ABE, AFK, being right angled at B and F, and having a common angle at A, are equiangular; therefore AB is to BE, as AF to FK, and AB to AE as AF to AK; but AB, AF are radiuses, and BE, FK tangents (Def. 8. *Pl. Tr.*), and AE, AK secants [Def. 9. *Pl. Tr.*], of the arches CB, CF, which are measures of the angle A.

Scholium.—The reader should beware, when the sines, tangents or secants of angles are mentioned, that he do not consider them, as right lines of a given length; but rather, as terms, denoting the ratios, which the right lines representing them have to radius, their length varying according to the radius to which they are referred, but the ratios between them are definite and fixed, which is sufficient for the purposes of trigonometry. They may be considered as the quotients, which arise from dividing the right lines which represent them, by the radius, which quotient would express their ratio to the radius, and would be a fixed quantity; but this way of conceiving the thing would rather belong to arithmetick than to geometry, would not be so easily understood, and would lead to nothing, which is not attainable in the usual way, being that which is in this tract pursued.

PROPOSITION I. THEOREM.

In a right angled plain triangle (ABC), the hypothenuse (AC) is to either of the legs, or sides including the right angle (as BC), as radius is to the sine of the angle (BAC) opposite that leg.



From the centre A, at the distance AC, let an arch of a circle CD be described, meeting AB produced in D; CB is the sine of the arch CD, or angle CAD (Def. 6. *Pl. Tr.*), and the ratio of radius to the sine of a given angle is invariable [Cor. 8. Def. *Pl. Tr.*]; therefore AC is to CB, as radius to the sine of the angle CAB.

In like manner, if a circle be described from the centre C, at the distance CA, meeting CB produced; it may be proved, that AC is to AB, as radius to the sine of the angle ACB.

PROP. II. THEOR.

In a right angled triangle (ABC, see fig. to preced. prop.), either of the legs (as AB) is to the other (BC), as radius is to the tangent of the angle (BAC) opposite that other, and to the hypothenuse (AC), as radius to the secant of the same angle.

From the centre A, at the distance AB, describe a circle, meeting AC in E; BC is the tangent, and AC the secant, of the arch BE, or angle EAB [Def. 8 and 9. *Pl. Tr.*], and the ratio of radius to the tangent or secant of a given angle is invariable [Cor. 8. Def. *Pl. Tr.*], therefore AB is to BC, as radius to the tangent, and to AC, as radius to the secant, of the angle CAB, opposite to BC.

In like manner, if a circle be described from the centre C, at the distance CB meeting CA; it may be proved, that BC is to AB, as radius to the tangent, and to AC, as radius to the secant, of the angle ACB, opposite to AB.

Scholium.—From this proposition and the preceding, arises that usual mode of speaking among mathematicians, that in a right angled plain triangle, if the hypothenuse be made radius, the legs become the sines of the opposite angles, and if either of the legs be made radius, the other leg becomes the tangent of the angle opposite to it, and the hypothenuse, the secant of the same angle.

PROP. III. THEOR.

The sides of plain triangles (as ABC, see figure 1, 2 and 3), are as the sines of the opposite angles.

Fig. 1.

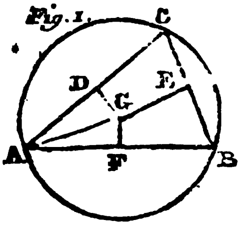


Fig. 2.

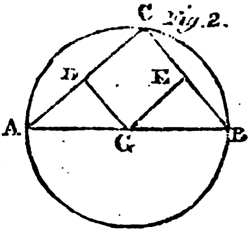
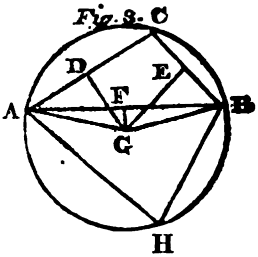


Fig. 3.



About the triangle ABC describe a circle, which is done as in 5. 4 Eu. by bisecting two of its sides AC and BC by perpendiculars DG and EG meeting each other in the centre G; and if, in the cases of fig. 1 and 3, GF be drawn perpendicular to AB, and AG be joined, AD becomes, in all the figures, the sine of the angle AGD, AG being radius [Def. 6. *Pl. Tr.*]; but AD is the half of AC (3. 3 *Eu.*), and the angle AGD is equal to ABC, being each of them half of the angle at the centre on the arch AC (20. 3 and 4. 1 *Eu.*); therefore the half of the side AC is equal to the sine of the angle ABC, AG being radius. In like manner it may be shewn, that the half of BC is the sine of the angle BAC, and, in fig. 1, that the half of AB is the sine of the angle C, to the same radius.

In a right angled triangle, fig. 2, AG the half of AB is radius, and therefore equal to the sine of the right angle ACB, [*Cor.* 2. *Def. Pl. Tr.*].

In an obtuse angled triangle fig. 3, join GB, AH and HB; AF the half of AB is the sine of the angle AGF [*Def.* 6. *Pl. Tr.*], and the angle AGF is the half of the angle AGB (4. 1 *Eu.*), and therefore equal to the angle H (20. 3. *Eu.*) but the angle H is the complement of the angle C to the two right angles (22. 3 *Eu.*), and the sine of an angle and of its complement to two right angles is the same [*Cor.* 1. *Def. Pl. Tr.*], therefore AF is the sine of the angle C. Thus, in every case, the halves of the sides becomes sines of the opposite angles, to the radius of the circumscribing circle, and the sides are as their halves (15. 5. *Eu.*), therefore, in every case, the sides are to each other, as the sines of the opposite angles.

Otherwise.

Fig. 1.

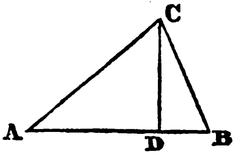
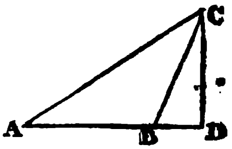


Fig. 2.



Let fall a perpendicular CD (see both figures), from an angle C, on the opposite side AB; and in the right angled triangle ADC, AC is to CD, as radius to the sine of A [1 *Pl. Tr.*], and CD is to CB, as the sine of the angle CBD to the radius (By the same); therefore, by perturbate equality, AC is to CB, as the sine of CBD to the sine of A (23. 5. *Eu.*) and therefore. in the case of fig. 2, the sines of the angles CBA and CBD being equal [*Cor.* 1. *Def. Pl. Tr.*], as the sine of the angle CBA to the sine of A. In like manner it may be proved, that any other two sides are to each other, as the sines of the opposite angles.

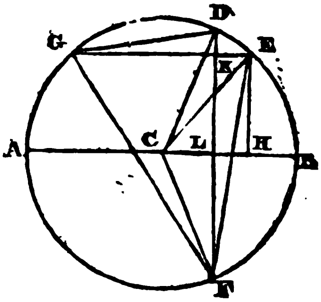
Otherwise.

The same construction remaining, the perpendicular CD being radius, the sides AC and CB are to each other, as the consecants of the adjacent angles at the base AB [2. *Pl. Tr. and* 22. 5 *Eu.*], or, the consecants of any two angles being inversely as their sines [*Cor.* 7. *Def. Pl. Tr.*], as the sines of the remote angles at the base, or, of the opposite angles.

PROP. IV. PROB.

The sum of the sines of any two arches (DB, EB) of a circle, or of any two angles (DCB, ECB), is to the difference of their sines, as the tangent of half the sum of the arches or angles, to the tangent of half their difference.

Let C be the centre of the circle, draw the diameter ACB. on which let fall the perpendiculars DL, EH, produce DL to meet the circumference again in F, draw EKG parallel to AB, and join CF, GD, GF and FE.



Because CL bisects the right line DF in L [3. 3. *Eu.*], and, because of the equality of the angles DCL and FCL, the arch DF in B [26. 3. *Eu.*], FL is equal to DL the sine of the arch DB, and EH or KL is the sine of the arch EB ; therefore FK is the sum of the sines of the arches DB and EB, and DK the difference of their sines: and, because the arches FB and DB are equal, FE is the sum, and DE the difference of the arches DB and EB, of which sum and difference the angles at the centre FCE and DCE are the measures [*Def* 2. *Pl. Tr.*], and therefore the angles FGK and DGK at the circumference, being the halves of these angles FCE and DCE at the centre [20. 3. *Eu.*], are the measures of half the same sum and difference ; but in the triangle FKG, right angled at K, FK is to KG, as the tangent of the angle FGK is to radius [2. *Pl. Tr. and Theor.* 3. 15. 5. *Eu.*], and in the triangle GKD, GK is to KD, as radius to the tangent of the angle DGK [2. *Pl. Tr.*]; therefore, by equality, FK is to KD, as the tangent of the angle FGK to the tangent of DGK [22 5. *Eu.*], or, which has been shewn to be equal, as the tangent of the half sum of the arches DB and EB, to the tangent of half the difference ; and it has been shewn, that FK and DK are the sum and difference of the sines of the arches DB and EB, which are therefore to each other in the ratio of that sum and difference [*Cor.* 1. 7. 5. *Eu.*]; therefore the ratios of that sum and difference, and of the tangents of the half sum and half difference of the same arches, being each equal to the ratio of FK to DK, are equal to each other [11. 5 *Eu*].

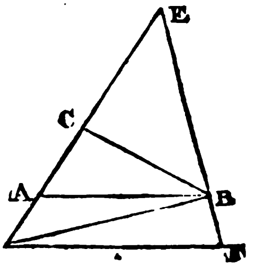
PROP. V. THEOR.

The sum of the cosines of any two arches (DB and EB, see prec. fig.), or of any two angles (DCB, ECB), is to the difference of their cosines, as the cotangent of half the sum of the arches or angles, to the tangent of half the difference.

The same construction, as in the preceding proposition, remaining, CL is the cosine of BD, and CH, equal to the half of GE, the cosine of EB; therefore GK is equal to the sum of these cosines, and KE to their difference; and it has been shewn in the preceding proposition, that the angle FGK, is the measure of the half sum of the arches DB and EB, and that DGK, and therefore EFK, which is equal to it [21. 3 *Eu.*], is the measure of half their difference; and, in the right angled triangle FKG, GK is to KF as the cotangent of the angle FGK is to radius [2. *Pl. Tr, Theor.* 3. 15. 5 *and Def.* 10. *Pl. Tr.*], and, in the triangle FKE, FK is to KE, as radius to the tangent of the angle EFK [2. *Pl. Tr.*]; therefore, by equality, GK is to KE, as the cotangent of the angle FGK to the tangent of EFK [22. 5 *Eu.*], or, as the cotangent of the half sum of the arches DB and EB, or angles DCB and ECB, to the tangent of half their difference; and it has been shewn, that GK and KE are the sum and difference of the cosines of these arches or angles to the radius CB, therefore GK and KE are to each other, in the ratio of that sum and difference [*Cor.* 1. 7. 5 *Eu.*]; and therefore the ratios of that sum and difference, and of the cotangent of the half sum of the same arches or angles to the tangent of half their difference, being each equal to the ratio of GK to KE, are equal to each other [11. 5 *Eu*].

PROP. VI. THEOR.

In a plain triangle (ABC) the sum of the legs (AC, CB) is to the difference of the legs, as the tangent of half the sum of the angles (CAB, CBA) at the base (AB) is to the tangent of half their difference.



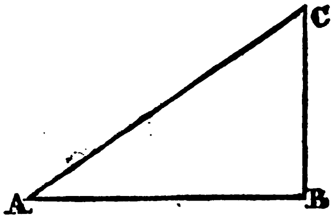
Let CB be the greater of the legs, and since CB is to CA as the sine of the angle CAB to the sine of CBA (3. *Pl. Tr.*), the sum of CB and CA is to their difference, as the sum of the sines of the angles CAB and CBA is to the difference of their sines (18, *Sch.* 18, and 22. 5 *Eu.*); but the sum of the sines of the angles CAB, CBA is to the difference of their sines, as the tangent of half their sum to the tangent of half their difference (4. *Pl. Tr*); therefore the ratios of the sum of CB and CA to their difference, and of the tangent of half the sum of the angles CAB and CBA to the tangent of half their difference, being each equal to the ratio of the sum of the sines of these angles to the difference of their sines, are equal to each other (11. 5 *Eu*).

Otherwise.

On the less of the legs AC, produced both ways, take CD and CE each equal to CB; join EB and DB, and through D, draw DF parallel to AB, meeting EB produced in F.

The two interior angles CDB and CBD of the triangle DBC, are equal to the exterior ECB (32. 1 *Eu.*), or, which is equal (by the same), to the two interior angles CAB and CBA of the triangle ABC; therefore CDB and CBD together are equal to the sum of the angles CAB and CBA at the base AB of the triangle ABC, and being, because of the equality of the sides CD and CB, equal to each other (5. 1 *Eu.*), one of them CDB is equal to half the sum of these angles.

And, because the exterior angle CAB of the triangle DBA is equal to the interior angles ADB and ABD, or, ADB being equal to CBD (5. 1 *Eu.*), to the angles CBD and ABD, or, to CBA and twice ABD, the angle ABD, or its equal (29. 1 *Eu.*), the alternate angle BDF, is half the difference of the angles CAB and CBA, at the base AB, of the triangle ABC.



Case.	Given.	Sought.	Solutions.
1.	Both legs. AB and BC.	The acute angles. A and C.	AB : BC : : R : T. of the angle A (2. *Pl.* Tr.), and the angle C is the complement of A to a right angle or 90°.
2.	The hypothenuse and a leg. AC & AB.	The acute angles.	AC : : AB : : R : S. C. (1. *Pl. Tr.*), and A is the complement of C to a right angle.
3.	A leg and the acute angles. AB and the angles A & C.	The other leg and the hypothenuse. BC and AC.	R : T. A : : AB : BC (2. *Pl. Tr.*). S. C. : R : : AB : AC (1. *Pl. Tr.*).
4.	The hypothenuse and the acute angles. AC and the angles A & C.	Either leg. AB or BC.	R : S. C. : : AC : AB (1. *Pl. Tr.*).

PROBLEM II.

Of the three sides and three angles of any plain triangle, any three being given, whereof one at least is a side, to find the rest: or, all the angles being given, to find the ratios of the sides.

If all the angles be given, the ratios of the sides are given, being as the sines of the opposite angles (3. *Pl. Tr.*).

Fig. 1.

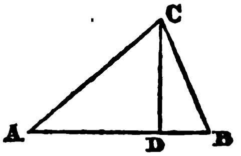
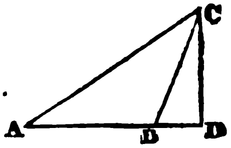


Fig. 2.



Case.	Given.	Sought.	Solutions.
1.	All the angles and one side. The angles A and ABC, and therefore ACB, and AB.	The other sides. AC & BC.	S. ACB : S. ABC : : AB : AC (3. *Pl. Tr*). Also S. ACB : S. A : :AB : BC (*By the same*).
2.	Two sides and an angle opposite to one of them. AC, BC and A.	The other angles. The angles ABC and ACB.	BC : AC : : S. A : S. ABC (3. *Pl. Tr*). If the side BC opposite the given angle be greater than the other given side AC, the angle ABC is acute. But if BC be less than AC, since the sine of an angle and of its complement to two right angles is the same, by cor. 1. Def. Pl. Tr.; the species of the angle B, namely, whether it be acute or obtuse, must be known, or the solution will be ambiguous. The angles A and ABC being given, the angle ACB may be found, being the complement of their sum to 180° or two right angles, by 32. 1 Eu.

Case.	Given.	Sought.	Solutions.
3.	Two sides and the included angle. AC, BC and the angle ACB.	The other angles. The angles A and ABC.	Let AC be the greater of the given sides, and $AC+BC : AC-BC :: T\,\frac{ABC+A}{2} : T\,\frac{ABC-A}{2}$ (6. Pl. Tr). Whence is found the difference of the angles A and ABC, whose sum is given, being the complement of the given angle ACB to two right angles, whence the angles A and ABC may be found by prop. 7 of this.
4.	All the sides. AB, BC and AC.	The angles.	Let a perpendicular CD be let fall from one of the angles ACB on the opposite side AB. And, by cor. 1. 5 and 6. 2 and 16. 6 Eu. in fig. 1 $AB : AC+CB :: AC-CB : AD-DB$, and in the case of fig. 2. $AB : AC+CB :: AC-CB : AD+BD$, whence the sum and difference of AD and BD being in either case given, the lines AD, BD themselves may be found by prop. 7 of this; and thence, by case 2 of right angled triangles, the angles CAD, CBD, and of course, in fig. 2. the angle ABC, the complement of CBD to two right angles by 13. 1 Eu., may be found.

SUPPLEMENT

TO THE SIX FIRST BOOKS OF

EUCLID'S ELEMENTS OF GEOMETRY.

BOOK I.

ELEMENTS OF CONICK SECTIONS.

Note.—*In subsequent citations, sup. denotes supplement.*

DEFINITIONS.

1. [*See note.*] *Conick Sections*, or *Pattalloids*, are figures formed by lines, the sums or differences of the distances of every point of which, from two given points, or the distances of every point of which from a given point, or from a given point and given right line, are equal. And these figures are sometimes, for brevity, called, sections.

Cor. 1. Right lines come within this definition.

For the sums of the distances of every point, as C and D (see fig. 1), in any right line AB, from its extremes A and B, are equal. And the differences of the distances of every point, as C and D (see fig. 2), in the production of any right line AB, from its extremes A and B, are equal.

Fig. 1.

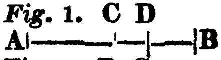
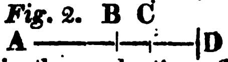
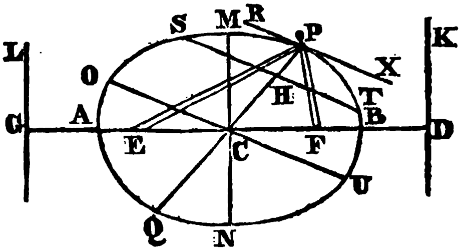


Fig. 2.





And every point, of a right line (MN), perpendicularly bisecting another right line (AB), is equally distant from the extremes (A and B), of the bisected line (4. 1 *Eu.*), and so comes within the definition.

Cor. 2. Circular lines come also within the definition.

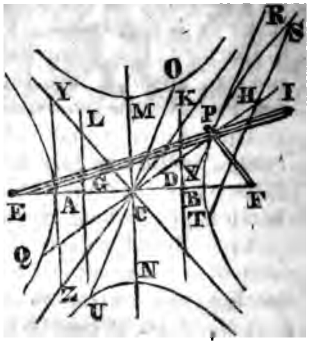
For the distances of every point thereof from a given point, namely, their centre, are equal (Def. 10. 1 *Eu*).

But, by the term, conick sections, are chiefly understood the figures, called Ellipses, Hyperbolas and Parabolas, which are defined in the 2d, 4th and 8th definitions following.

2. An *Ellipse*, is a conick section, bounded by a line (AMBN), the sums of the distances of every point of which, from two given points (E and F) within the same, are equal.

3. These two given points are called, the *focuses;* the point (C), wherein the right line (EF), joining the focuses, is bisected, the *centre* of the ellipse; any right line (QP), passing through the centre, and terminated both ways by the ellipse, a *diameter;* the points (Q and P), wherein it meets the ellipse, the *vertices* of the diameter; the diameter (AB) which passes through the focuses, the *greater, transverse* or *principal axis,* and its vertices (A and B), the *principal vertices;* that diameter (MN), which is at right angles thereto, the *less* or *second axis.* The distance (CE or CF) of the centre from either focus, the *eccentricity* of the ellipse.

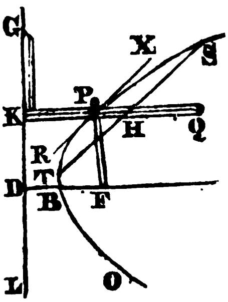
4. A *Hyperbola*, is a conick section, formed by a line (AQ or BP), the differences of the distances, of every point of which from two given points, (E and F) on different sides thereof, are equal; and if from the two points (A and B), in the right line (EF) joining these given points, the difference of whose distances from the same points is equal to the given difference of distances, two such figures (AQ and BP) be formed, these figures are called, *opposite hyperbolas.*



5. The two given points (E and F) are called, the *focuses;* the point (C), wherein a right line joining them is bisected, the *centre* of the hyperbola or opposite hyperbolas; any right line (QP), passing through the centre, and terminated both ways by the opposite hyperbolas, a *transverse diameter;* the points (Q and P), wherein it meets the hyperbolas, the *vertices* of the diameter; the diameter (AB), which produced passes through the focuses, the *transverse* or *principal axis,* and its vertices (A and B), the *principal vertices;* the right line, which passes through the centre, at right angles to the transverse axis, and the distance of either extreme of which from either of the principal vertices, is equal to that of either focus from the centre, the *second axis.* The distance (CE or CF), of the centre from either focus, the *eccentricity* of the hyperbola or opposite hyperbolas.

6. Four hyperbolas (AQ, BP, MO, NU) are said to be *conjugate;* when the transverse axis of two of them, is the second axis of the other two, and the contrary.

7. Any right line (OU), passing through the centre, and terminated both ways, by hyperbolas conjugate to those which pass through the principal vertices, is called, a *second diameter.*



8. A *Parabola,* is a conick section, formed by a line (BPS), the distances of every point of which, from a given point (F), and a given right line (DK), are equal.

9. That given point (F), is called the *focus,* and that given right line (DK), the *directrix* of the parabola; every right line (as KPQ), perpendicular to the directrix, is called, a *diameter;* the point (P), wherein it meets the parabola, the *vertex* of that diameter; the diameter (DBF), which passses through the focus, the *axis* of the parabola; and its vertex, (B), the *principal vertex.*

10. A right line (RPX, see the 3 prec. fig.), which meets a conick section in any point (P), and, being produced both ways, falls wholly without the section, is said to be a *tangent* to the section, or to *touch* it in that point.

11. But if a right line, meeting a conick section, is on one side of its concourse with the section, within, and on the other, without the section, it is called, a *secant.*

12. A right line (ST), drawn from any point (S) of a conick section, meeting a diameter (PQ) of the section, and parallel

to a tangent (RPX) to the section, passing through the vertex (P) of the diameter, is said to be *ordinately applied* to that diameter; and the part (SH or TH), of the right line so applied, between the section and the diameter, is said to be an *ordinate* to the diameter.

And a right line, terminated by opposite hyperbolas, is said to be *ordinately applied* to the diameter, whose vertex is in the contact, of a tangent, parallel to the right line so terminated.

Scholium. Hence, in the circle, perpendiculars let fall from the circumference to a diameter, are *ordinates* to that diameter.

13. A segment (PH), of a diameter (PQ), between an ordinate thereto, and a vertex of the diameter, is called an *abscissa*.

14. Two diameters of an ellipse or hyperbola, each of which is parallel to a tangent, passing through a vertex of the other, are called, *conjugate diameters.*

From the 12th definition it is manifest, that either of two conjugate diameters is ordinately applied to the other.

15. A right line, which is a third proportional to two conjugate diameters of an ellipse or hyperbola, is said to be, the *parameter* or *latus rectum,* of that diameter, which is the first of the three proportionals.

16. A right line, which is four-fold the distance of the vertex of a diameter of a parabola, either, from the directrix or the focus, is said to be, the *parameter* or *latus rectum,* of that diameter.

17. The parameter, which belongs to the axis of a parabola, or the transverse axis of an ellipse or hyperbola, is said to be the *principal parameter* or *latus rectum* of the section.

18. A right line perpendicularly cutting the transverse axis of an ellipse or hyperbola, produced in the ellipse; and whose distance from the centre, is a third proportional, to the eccentricity and the transverse semiaxis, is called a *directrix* of the section.

19. Right lines (CY and CZ, See fig. to Def. 4.), passing thro' the centre (C) of a hyperbola, and the extremes of a right line (YZ) equal to the second axis (MN), and perpendicularly bisected by the transverse axis (AB) in a vertex (A), are called, *asymptotes.*

Cor.—The angle (YCZ) formed by the asymptotes (CY, CZ) towards either of the opposite hyperbolas (as AQ) is bisected by the axis (AB).

For, in the triangles CAY, CAZ, the sides AY and AZ are equal, being each equal to CM (by this def.), AC common, and the angles at A equal, being right angles; therefore the angle ACY is equal to ACZ [4. 1. *Eu.*]

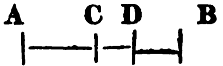
20. Hyperbolas are said to be, *right angled*, when the asymptotes are at right angles to each other.

Schol.—That, in this case, any two conjugate diameters are equal, is demonstrated in prop. 54 of this: whence the hyperbolas in this case are also said to be equilateral, as is in that proposition observed.

21. Two conick sections, which touch the same right line in the same point, are said to touch each other in that point.

22. A circle, which so touches a conick section in any point, that, between it and the section, no other circle, described through that point, can pass, is said, *to have the same curvature* with the section in the point of contact, or to *osculate* the section in that point.

23. If the ratio of the principal axis of an ellipse or hyperbola to its second axis, be the same, as that of the principal to the second axis, of another ellipse or hyperbola, these two ellipses or hyperbolas are said to be *similar*.



24. If a right line (AB) be so divided in two points (C and D), that the whole (AB) is to either extreme part (AC), as the other extreme part (DB) is to the middle part (CD), that right line (AB) is said to be, *harmonically divided*.

POSTULATES.

1. *That, from any two given points as focuses, and through any other given point not in the right line joining them, an ellipse may be described.*

The genesis or formation of the ellipse may be thus conceived. Let E and F (see figure to Def. 2. above), be the points from which, as focuses, and P the point, through which, the ellipse is to be described. Let the extremities E and F of a thread or flexible line EPF (whose length is equal to the two right lines EP and PF, be fastened in the points E and F, and by means of a pin P, let the thread be extended, and the pin P be moved round, the thread remaining continually extended, till it return to the place from which it began to move; then is the sum of the distances of every point of the line described by the pin P, from the points E and F, always equal, as is manifest, and therefore that line will form an ellipse described from the points E and F as focuses, and through the point P (by Def. 2. 1. Sup).

2. *That, from any two given points as focuses, and through any other given point neither in the production of a right line joining them, nor in a right line perpendicularly bisecting them, a hyperbola and its opposite may be described.*

The genesis or formation of the hyperbola or opposite hyperbolas may be thus conceived. Let E and F (see fig. to Def. 4. above), be the two points, from which, as focuses, and P the point, through which, the hyperbola is to be described, the description of its opposite being also required.

Let one extreme of a ruler EI, be so affixed at the point E, that it may be freely moved round that point as a centre; let the ruler be so placed, as to pass by the point P, through which the figure is to be described, let a pin be so fixed in the ruler, as to be moveable to and from the point I at pleasure, and let one extreme of a thread or flexible line FPI, whose length is equal to the right lines FP and PI together, be affixed at the point F, and go close round the pin in the situation P, so that the pin may be within the angle FPI, let its other extremity be affixed to the extremity of the ruler, and let the ruler be moved round the point E, towards S or T, and, the thread or line still remaining extended, let the pin, affixed to the side of the ruler, describe the line BPS; and since the point P is neither in the production of the right line EF, nor in a right line, as MN perpendicularly bisecting it, it is manifest, that the difference of EP and PF is always, during the above description, equal to a given right line, and because EF and FP together are greater than EP (20. 1 *Eu.*), taking FP from each, EF is greater than the excess of EP above PF, or than the given difference of EP and PF (*Ax.* 5. 1 *Eu.*), and therefore the line described by the pin intersects the right line EF somewhere between E and F as in B, (for if it could intersect the right line EF produced, EF would be equal to the given difference of EP and PF, contrary to what has been proved), whence, the points E and F being on different sides of the described line BPS, and the difference of the distances of every point of that line from the same points E and F always equal to a given right line, that described line BPS is a hyperbola, described from the points E and F as focuses, and through the point P (by Def. 4. 1 Sup).

Let now EA be taken on EF equal to BF, and the extreme of the ruler, which was before affixed at the point E, be now affixed at the point F, and a hyperbola QA be described in like manner as above, from the points E and F as focuses, through the point A; and since EA is equal to BF, the difference of EF and EA is equal to the difference of EF and BF, and therefore the hyperbolas AQ and BP described from the focuses E ond F, are opposite hyperbolas (Def. 4. 1. Sup). And these lines may be extended to a distance from the points E and F, greater than any given distance, namely, if a thread be taken, whose length is greater than that distance.

3. *That to a given right line, as directrix, and from any given point not therein, as focus, a parabola may be described.*

The genesis or formation of the parabola may be thus conceived. Let LG (See fig. to Def. 8. above), be the right line, to which, as directrix, and F the point, from which, as focus, the parabola is to be described: let one side KG, of a square GKQ, be so applied to the directrix LG, that the side KQ, may pass through the point F, and the point K may coincide with the point D; bisect the right line DF in B, and to the extremity Q, of the side KQ, let one extremity of a thread of the same length as KQ be fastened, and let its other extremity, the thread going round a pin, in the side KQ of the square at the point B, be fastened at the point F, which it is manifest, it would reach, because DB and BF are equal [Constr.]; let the side KG of the square be moved along the right line LG, and, the thread remaining extended, let the pin, affixed to the side KQ of the square, describe the line OPS. And since, in every situation of the pin, during the description of the line OPS, the side KQ is equal in length to the thread FPQ, taking from each PQ, which is common, KP is equal to PF [Ax. 3. 1. Eu.]; and so the distances of every point of the line OPS from the point F and right line LG are equal, therefore the line OPS is a parabola (Def. 8. 1. Sup). And this line may be extended to a distance from the point F greater than any given distance, namely, if a square be taken, the length of whose side KQ is greater than that distance.

PROPOSITION I. THEOREM.

In ellipses (as APB, fig. 1), the sum, and in hyperbolas (as AQ, BP, fig. 2), the difference, of the distances (EP, PF) of any point (P) of the section, or opposite sections, from the focuses (E and F), is equal to the principal axis (AB); and either axis (AB or MN) is bisected in the centre (C).

Fig. 1.

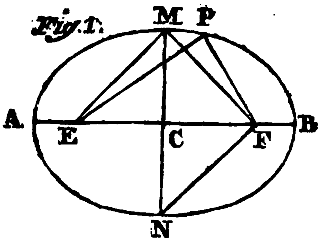
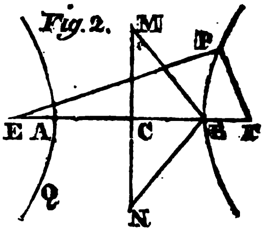


Fig. 2.



Part 1. In the ellipse (see fig. 1), the sum of AE and AF, or of twice AE and EF, is equal to the sum of FB and EB (*Def.* 2. 1 *Sup.*), or of twice FB and EF; taking away EF which is common, twice AE is equal to twice FB, and so AE to FB (*Ax.* 7. 1 *Eu.*); therefore AF and AE together are equal AF and FB together, or AB; but EP and PF together are equal to AF and AE together (*Def.* 2. 1 *Sup.*), therefore EP and PF together, are equal to AB.

Part 2. In like manner, in the hyperbola (see fig. 2), the difference of AF and AE, or of EF and twice AE, is equal to the difference of EB and FB (*Def.* 4. 1 *Sup.*), or of EF and twice FB, wherefore, taking from EF that difference, the remainder is equal to either twice AE or twice FB, which are therefore equal to each other (*Ax.* 3. 1 *Eu.*), and so AE is equal to FB (*Ax.* 7. 1 *Eu.*); therefore the difference of AF and AE, is equal to the difference of AF and FB, or AB; but the difference of EP and PF is equal to the difference of AF and AE (*Def.* 4. 1 *Sup.*); therefore the difference of EP and PF is equal to AB.

Part 3. And, AE having been proved equal to BF, and EC being equal to CF (*Def.* 3 *and* 5. 1 *Sup.*), AC is equal to CB (*Ax.* 2 *and* 3. 1 *Eu.*), and so AB is bisected in C.

And in fig. 1, EM and MF being joined, the triangles ECM and FCM, having CE and CF equal (*Def.* 3. 1 *Sup.*), CM common, and the angles at C right (by the same), ME and MF are equal (4. 1 *Eu.*); whence, EM and MF together being equal to AB (part 1, of this), either of them, as FM, is equal to its half CB; in like manner FN may be proved equal to CB; therefore FM and FN are equal, and of course the angles FMN and FNM (5. 1 *Eu.*), whence, the triangles MCF and NCF having also the right angles at C equal, MC is equal to CN (26 1. *Eu.*), and so MN is bisected in C.

In fig. 2, MB and BN, being drawn, are equal (*Def.* 5. 1 *Sup.*), and therefore the angles BMN and BNM (5. 1 *Eu.*); whence, the triangles MCB and NCB, having also the right angles at C equal, MC is equal to CN (26. 1 *Eu.*), and so MN is bisected in C.

Cor. "The distance of a vertex (M, see fig. 1), of the second "axis (MN) of an ellipse from either focus (E or F), is equal "to the principal semiaxis." It having been proved, in the demonstration of part 3 of this proposition, that either EM or MF is equal to CB.

PROP. II. THEOR.

The square of the second semiaxis (CM, see fig. to prec. prop.), of an ellipse or hyperbola (BP), is equal to the difference of the squares of the principal semiaxis (CB), and the eccentricity (CF), or to the rectangle under the distances of either focus (as F), from the principal vertices (A and B), or of either principal vertex (as B), from the focuses (E and F).

In the ellipse (see fig. 1), draw MF, which is equal to CB [*Cor.* 1. 1 *Sup.*]; whence, in the triangle CFM, right-angled at C, the square of CM is equal to the difference of the squares of MF and CF [47. 1 *Eu.*], or of CB and CF, or, which is equal [*Schol.* 6. 2 *Eu.*], to the rectangle AFB or EBF.

In the hyperbola (see fig. 2), draw MB, which is equal to CF [*Def.* 5. 1. *Sup.*]; whence, in the triangle CBM, right angled at C, the square of CM is equal to the difference of the squares of MB and CB [47. 1 *Eu.*], or of CF and CB, or, which is equal [*Schol.* 6. 2 *Eu*[, to the rectangle AFB or EBF.

Cor. 1. Hence, if, in ellipses, there be taken two points (E and F, see fig. 1), in the principal axis (AB), the rectangle under the distances of each of which from the principal vertices (A and B, namely, the rectangle AEB or AFB), is equal to the square of the second semiaxis (CM), which may be done by cor. 2. 5 and 6. 2 Eu. these points are the focuses of the ellipse.

Cor. 2. In like manner, if, in hyperbolas, there be taken two points (E and F, see fig. 2), in the principal axis (AB) produced both ways, the rectangle under the distances of each of which from the principal vertices (A and B, namely, the rectangle AEB or AFB), is equal to the square of the second semiaxis (CM), which may be done by cor. 3. 5 and 6. 2 Eu. these points are the focuses of the hyperbola or opposite hyperbolas.

PROP. III. THEOR.

The sum of the distances (GE and GF, see fig. 1), of any point (G) without an ellipse (APB), from the focuses (E and F), is greater, of any point (H) within it, less, than the principal axis (AB).

And the difference of the distances (GE and GF, see fig. 2), of any point (G) without opposite hyperbolas (AO and BP), from the focuses (E and F), is less, of any point (H), within one of them (BP), greater, than the principal axis (AB).

And the distance (GF, see fig. 3), of any point (G) without a parabola (KP), from the focus (F), is greater, of any point (H) within it, less, than the distance of the same point from the directrix (DO).

Fig. 1.

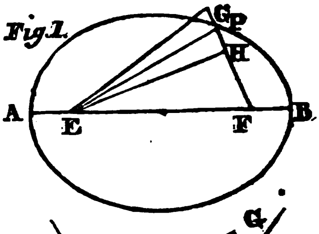


Fig. 2.

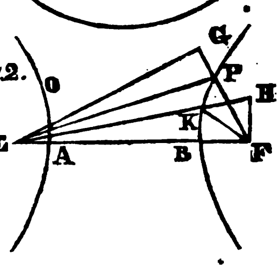
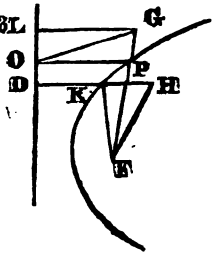


Fig. 3.



Part 1, *fig* 1. Let FG, or FH produced, meet the ellipse in P; and EG and GF together, are greater, and EH and HF together, less, than EP and PF [21. 1 *Eu.*], or, which is equal [1. 1 *Sup*], AB.

Part 2, *fig.* 2. Let GF meet the hyperbola, BP in P, and because EG is less than EP and PG together [20. 1 *Eu.*], the excess of EG above GF, is less than the excess of EP and PG together above GF [*Ax.* 5. 1 *Eu.*], or, of EP above PF, or [1. 1 *Sup.*], AB.

And the difference of EH and HF is greater than AB.

For, having drawn HE meeting the hyperbola BP in K, and joined KF; because HF is less than HK and KF together [20. 1 *Eu.*], the excess of EH above HF, is greater than the excess of EH above HK and KF together, or of EK above KF, or [1. 1 *Sup.*], AB.

Part 3, *fig.* 3. Join GF and HF, and draw GL and HD perpendicular to DO; and, because G is without the parabola and F within it, GF meets the parabola, as in P, and, for the same reason, HD meets it, as in K; draw PO perpendicular to DO, and join KF and GO; and PF and PO being equal [*Def.* 8. 1 *Sup.*], GF is equal to GP and PO together [*Ax.* 2. 1 *Eu*], and therefore, GP and PO together being greater than GO [20. 1 *Eu.*], and GO than GL [*Cor.* 1. 19. 1 *Eu.*], GF is greater than GL. And HF is less than HK, KF together [20. 1 *Eu.*], or, KB being equal to KF [*Def.* 8. 1 *Sup.*], than HD.

PROP. IV. THEOR.

If the sum of the distances of any point from the focuses (E and F, see fig. 1 above), of an ellipse, be equal to the principal axis (AB), that point is in the ellipse; if that sum be greater, it is without, if less, within the same.

If the difference of the distances of any point from the focuses (E and F, see fig. 2 above), of a hyperbola, be equal to the principal axis, that point is in one of the opposite hyperbolas; if that difference be less, it is without both, if greater, within one of them.

And if the distance of any point from the focus (F, see fig. 3 above), of a parabola, be equal to its distance from the directrix (DO), that point is in the parabola; if the distance from the focus be greater, the point is without, if less, within it.

Part 1. *Fig.* 1. If the sum of EP and PF be equal to AB, P is in the ellipse; for, if P were without the ellipse, that sum would be greater, if within it, less, than AB [3. 1 *Sup.*], contrary to the supposition.

If the sum of EG and GF be greater than AB, G is without the ellipse; for, if G were in the ellipse, that sum would be equal to, if within it, less than AB [1 and 3. 1 *Sup*], contra hyp.

And if the sum of EH and HF be less than AB, H is within the ellipse; for, if H were in the ellipse, that sum would be equal to, if without it, greater than AB (1 and 3. 1 *Sup.*), contra hyp.

Part 2, *fig.* 2. If the difference of EP and PF be equal to AB, P is one of the opposite hyperbolas; for, if P were without both of them, that difference would be less, if within one of them, greater, than AB (3, 1 *Sup.*), contra hyp.

If the difference of EG and GF be less than AB, G is without both of the opposite hyperbolas; for, if G were in one of these hyperbolas, that difference would be equal to, if within one of them, greater than, AB (1 and 3. 1 *Sup.*), contra hyp.

And if the difference of EH and HF be greater than AB, H is within one of the opposite hyperbolas ; for, if H were in one of them, that difference would be equal to, if without both of them, less than, AB (1 and 3. 1 *Sup.*), contra hyp.

Part 3, *fig*. 3. If PF drawn to the focus be equal to PO drawn perpendicularly to the directrix, P is in the parabola ; for, if P were without the parabola, PF would be greater, if within it, less than PO (3. 1 *Sup.*), contra hyp.

If GF drawn to the focus be greater than GL drawn perpendicularly to the directrix, G is without the parabola ; for, if G were in the parabola, GF would be equal to, if within it, less than, GL (1 and 3. 1 *Sup.*), contra hyp.

And if HF drawn to the focus be less than HD drawn perpendicularly to the directrix, H is within the parabola ; for, if H were in the parabola, HF would be equal to,. if without it, greater than, HD (1 and 3. 1 *Sup.*) contra hyp.

PROP. V. THEOR.

Any diameter of an ellipse or hyperbola is bisected in the centre.

Fig. 1

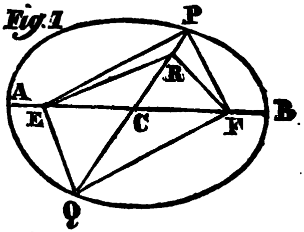
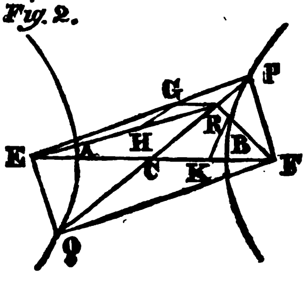


Fig. 2.



If the diameter be an axis, the proposition is demonstrated in 1. 1 Sup.

But if it be any other diameter, as QP, (see fig. 1 and 2), C being the centre, QP is bisected in C.

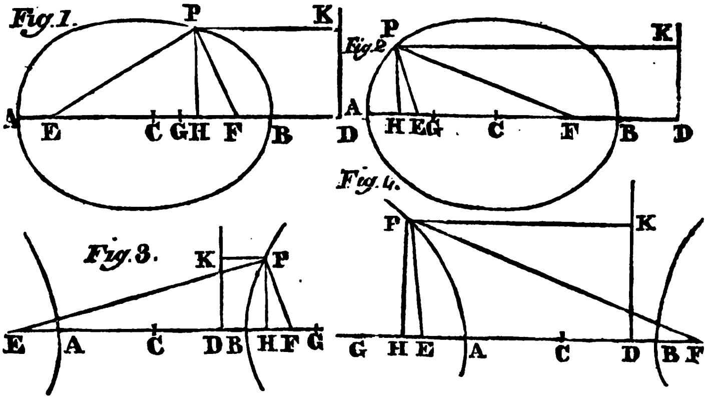
For if QC and CP be not equal, let one of them, as CP, if possible, be greater than the other QC, and take on CP, a part CR equal to CQ, and to the focuses E, F, draw PE, PF, QE, QF, RE and RF, and join EF, producing it, in fig. 1, both ways, to the principal vertices A and B.

The triangles ECR, QCF (see both fig.), having the sides EC, CR and angle ECR, severally equal to FC, CQ and the angle FCQ, the right lines ER and FQ are equal [4. 1 *Eu.*]; for the same reason, FR and EQ are equal, therefore the sum in fig. 1, and difference in fig. 2, of ER and RF is equal to the sum or difference, as the case may be, of FQ and QE, or, which is equal [1. 1 *Sup.*] of EP, PF. But this is absurd, in fig. 1, ER, RF being less than EP, PF, [21. 1 *Eu.*], therefore CP is not greater than CQ in that case; in like manner it may be proved, that CP is not less than CQ, therefore CP and CQ are equal, and QP is bisected in C, in the case of the ellipse, fig. 1.

And in the case of the hyperbola, fig. 2, besides the above construction, on PE take PG equal to PF, and on RE, RH equal to RF; join RG and GH, divide EF in K, so that EK may be to KF, as EP to PF [*Cor.* 1. 10. 6 *Eu.*], and join PK: and, since the triangles ECP and FCP have the sides EC and CP severally equal to FC and CP, but the obtuse angle ECP is greater than the acute angle FCP, the base EP is greater than the base FP [24. 1 *Eu.*], therefore EK is greater than KF [*constr.* and *cor.* 13, 5 *Eu.*], therefore EC and CF being equal [*Def.* 5. 1 *Sup.*], the point K falls between C and F; and, because EP is to PF, as EK to KF [*constr.*], the angle EPK is equal to KPF [3. 6 *Eu.*], therefore the angle EPC is less than CPF; whence, the triangles GPR and FPR having the sides GP and PR, severally equal to FP and PR, but the angle GPR less than FPR, the base GR is less than the base RF [24. 1 *Eu.*], or its equal RH; therefore the angle RHG is less than RGH [18. 1 *Eu.*], and therefore acute [32. 1 *Eu.*], and so the angle EHG is obtuse [13. 1 *Eu.*], and therefore EGH acute [32. 1 *Eu.*]; therefore EG, the excess of EP above PF, is greater than EH, the excess of ER above RF [19. 1 *Eu.*]; but these excesses are above proved to be equal, which is absurd; therefore CP is not greater than CQ; in like manner it may be proved, that it is not less, therefore CP and CQ are equal, and QP is bisected in C, in the case of the hyperbola, fig. 2.

PROP. VI. THEOR.

The ratio of the distance, of any point of a conick section from a focus, to the distance, of the same point from the directrix adjacent to that focus, is always the same, whatever point of the section be taken.



First, let the figure be an ellipse or hyperbola, and let PF (see all the four figures), be a right line drawn from any point P of the section, to a focus F, and PK, a perpendicular, let fall from P, on the directrix DK adjacent to that focus; let AB be the principal axis, and C the centre, of the section. The ratio of PF to PK is the same, wherever in the section the point P be taken.

Let E be the other focus of the section, join EP, let fall the perpendicular PH on AB, and from B, the principal vertex adjacent to F, towards H, take BG equal to PF.

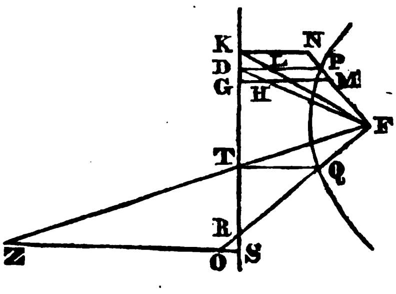
The rectangle under the sum and difference of EP and PF, is equal to the rectangle under the sum and difference of EH and HF [*Cor.* 1. 5 *and* 6. 2 *Eu.*], whence the rectangles under the half sums and half differences of these right lines, being similar to those under their sums and differences [*Def.* 1. 6 *Eu.*], are to each other in the same ratio, as the rectangles under those sums and the differences [15. 5 and 22. 6 *Eu.*], which ratio being that of equality, the rectangle under the half sum and half difference of EP and PF, is equal to the rectangle under the half sum and half difference of EH and HF; but, in the ellipse, CB is half the sum, and CG half the difference, and, in the hyperbola, CB is half the difference, and CG half the sum, of EP and PF [1. 1 *Sup. and constr.*], and, when the point H falls between E and F, as in fig. 1 and 3, CF is half the sum, and CH half the difference; and, when otherwise, as in fig. 2 and 4, CF is half the difference, and CH half the sum, of EH and HF [*Def.* 3 and 5. 1 *Sup.*]; therefore, in every case, the rectangle under CB and CG is equal the rectangle under CF and CH, and so CG is to CH, as CF to CB [16. 6 *Eu.*], or, which is equal [*Def.* 18. 1 *Sup.*], as CB to CD; and, by alternating, CG is to CB, as CH is to CD [16. 5 *Eu.*]; therefore, in the cases of fig. 2 and 4, the sum, and in those of fig. 1 and 3, the difference, of CG and CB, or BG, or its equal [*constr.*] PF, is to CB, as HD, the sum or difference, as the case may be, of CH and CD, or its equal [34. 1 *Eu.*] PK, is to CD [17 and 18. 5 *Eu.*]; and, by alternating, PF is to PK, as CB is to CD [16. 5 *Eu.*], or, which is equal [*Def.* 18. 1 *Sup.*], as CF is to CB, and therefore in a constant ratio, and always the same, wherever in the section the point P be taken.

In the case of a parabola, the ratio of these distances is always the same, being the ratio of equality [*Def.* 8. 1 *Sup*].

Scholium. This ratio is called *the determining ratio* of the section, and, in the ellipses and hyperbolas, is the same, as that of the eccentricity (CF) to the principal semiaxis (CB).

PROP. VII. THEOR.

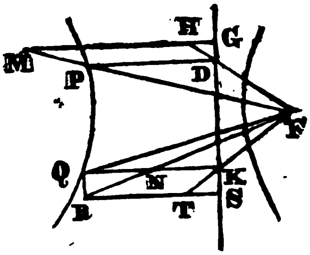
The ratio, of the distance, of any point within a conick section, or either of the opposite sections, from a focus, to its distance from the directrix adjacent to that focus, is less, of any point without, greater, than the determining ratio.



First. Let the point M be within the section, F being the focus within the same, and SK the adjacent directrix, let FM produced meet the section in P, and let fall the perpendiculars PD and MG on SK, draw DF meeting MG in H ; and, because of the equiangular triangles FMH and FPD, FM is to MH, as FP to PD [4. 6 *Eu.*], but the ratio of FM to MG is less than that of FM to MH [8. 5 *Eu.*], and therefore than that of FP to PD, or the determining ratio.

Secondly. Let the point N be without the section, being on the same side of the directrix, as the focus F, join NF meeting the section in P, let fall the perpendicular NK on DG, join KF, meeting PD in L ; and, because of the equiangular triangles FNK and FPL, FN is to NK, as FP is to PL [4. 6 *Eu.*], but the ratio of FP to PL is greater than of FP to PD [8. 5 *Eu.*], therefore the ratio of FN to NK is also greater than of FP to PD, or the determining ratio.

Thirdly. Let the point O be on the other side of the directrix SK of an ellipse or parabola, with respect to the focus F ; join FO, cutting the section in Q, and the directrix in R ; draw OS and QT perpendicular to the directrix, join FT, which produce to meet SO produced in Z ; and, because of the equiangular triangles ZOF and TQF, FO is to OZ, as FQ is to QT ; but QT is not less than FQ [*Def.* 8 and *Cor.* 6. 1 *Sup.*], therefore ZO is not less than OF [*Cor.* 13. 5 *Eu.*], but OF is greater than OS [9 *Ax. and Cor.* 19. 1 *Eu.*], therefore ZO is greater than OS, and so the ratio of FO to OS is greater than of FO to OZ [8. 5 *Eu.*], or, because of the equiangular triangles ZOF and TQF, of FQ to QT, or the determining ratio.



Fourthly. Let M be within the opposite hyperbola. Join FM cutting the opposite hyperbola in P, let fall the perpendiculars MG and PD on the directrix SK, join FD, which produce to meet MG, as in H; and, because of the equiangular triangles FMH, FPD, FM is to MH, as FP to PD [4. 6 *Eu.*], but the ratio of FM to MG is less than of FM to MH [8. 5 *Eu.*], and therefore than of FP to PD, or the determining ratio.

Lastly. Let the point N be taken any where between the directrix SK and the opposite hyperbola. Draw NK perpendicular to SK, which produce to meet the opposite hyperbola in Q; join FQ, and draw QR parallel to the directrix, meeting FN produced in R; draw RS parallel to QK, join FK, and produce it to meet RS in T. And, because the angle FRQ belonging to the right angled triangle NQR is acute, and FQR not acute, FR is greater than FQ [19. 1 *Eu.*]; but, because of the equiangular triangles FNK and FRT, the ratio of FN to NK is the same, as of FR to RT [4. 6 *Eu.*], and therefore [8. 5 *Eu.*] greater than that of FR to RS or QK, or, [8. 5 *Eu.*], of FQ to QK, or the determining ratio.

PROP. VIII. THEOR.

If the distance of any point from a focus of a conick section, be to its distance from the directrix adjacent to that focus, in the determining ratio, that point is in the section, or, in the case of a hyperbola, in one of the opposite sections; if in a less ratio, than the same determining one, the point is within, if in a greater, without the section or sections, as the case may be.

Part 1. Let the distance FP (see 1st fig. of preceding prop.), of any point P, from the focus F of a conick section, be to its distance PD from the directrix SK adjacent to that focus, in the determining ratio. P is in the section, or, in the case of a hyperbola, in one of the opposite sections.

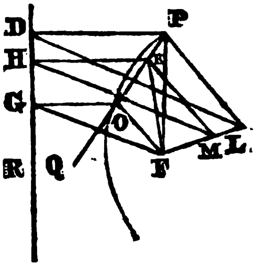
For, if P be not in the section, it must be either within, or without the section or sections; it is not within, for if it were, the ratio of FP to PD would be less than the determining ratio [7. 1 *Sup.*], contrary to the supposition; neither is it without, for then the ratio of FP to PD, would be greater than that ratio [7. 1 *Sup.*], which is also contrary to the supposition.

Part 2. Let now the ratio of the distances of any point, as M, from the focus and adjacent directrix, be less than the determining ratio, that point is within the section, or one of the opposite sections; for if it were in the section or its opposite, the ratio of the distance of that point from the focus to its distance from the directrix, would be equal to, if without the section, or in the case of a hyperbola, both of the opposite sections, greater than, the determining ratio [6 and 7. 1 *Sup.*], both of which are contrary to the supposition.

Part. 3. Let lastly the ratio of the distances of any point, as N, from the focus and adjacent directrix, be greater than the determining ratio, that point is without the section or opposite sections; for if were in the section or its opposite, the ratio of the distance of that point from the focus to its distance from the directrix, would be equal to, if within the section, or, in the case of a hyperbola, either of the opposite sections, less than, the determining ratio [6 and 7. 1 *Sup*], both of which are contrary to the supposition.

PROP. IX. THEOR.

Every right line, joining two points of a conick section, falls wholly within the section.



Let OP be a right line joining two points O and P of a conick section. OP falls wholly within the same.

Let F be a focus of the section in which the points O and P are and DR the directrix adjacent thereto. Take any point whatever K in OP; join OF, KF and PF, and, on the directrix DR, let fall the perpendiculars OG, KH and PD; draw PL parallel to OF, and equal to PF, join FL, through K, draw KM parallel to OF or PL, meeting FL in M, and join DL, HM and GF, and produce PO, as to Q.

Because of the right angles OGR, PDR, GO is parallel to DP (28. 1 *Eu.*], therefore the external angle GOQ is equal to the internal remote DPO (29. 1 *Eu.*); for a like reason, the angle QOF is equal to OPL; therefore the whole angle GOF is equal to the whole angle DPL (*Ax.* 2. 1 *Eu.*); and GO is to OF, as DP is to PF (6. 1 *Sup.*), or its equal (*Constr.*) PL; therefore the triangles GOF and DPL are equiangular (6. 6. *Eu.*), and the angles OGF and PDL equal, which being taken from the right angles OGR and DPR, the remaining angles FGR and LDR are equal, and therefore GF and DL parallel (28. 1 *Eu*).

And because of the parallels DP, HK and GO, the right line GH is to HD, as OK is to KP, for if GD and OP were parallel, OK and KP would be equal to GH and HD (34. 1 *Eu.*), and therefore in the same ratio, as these right lines (*Cor.* 2. 7. 5 *Eu.*), and if GD and OP were not parallel, they would meet, and, with a right line joining their remote extremes, form a triangle, and then GH would be to HD, as OK to KP (*Cor.* 2. 10. 6 *Eu.*); for a like reason, because of the parallels OF, KM and PL, FM is to ML, as OK to KP; therefore GH is to HD, as FM is to ML (11. 5 *Eu.*); whence, GF being parallel to DL, HM is parallel to each, for if it were not, a right line drawn through its intersection either with GD or FL, as through H, parallel to DL, would meet the other FL, in a point different from M, and, by a similar reasoning to that used above about GD and OP, might be shewn, to cut the right lines GD and FL proportionally, therefore the ratio of FM to ML would not be equal to that of GH to HD (8 and 13. 5 *Eu.*), contrary to what has been just proved, therefore HM is parallel to GF or DL.

And the angle HKM may in like manner, as GOF has been, be proved equal to DPL, and because HM is parallel to DL, the external angle MHR is equal to the internal remote LDR (29. 1 *Eu.*), which being taken from the right angles KHR and PDR, the remaining angles KHM and PDL are equal, therefore the triangles HKM and DPL are equiangular (32. 1 *Eu.*), and so KM is to KH, as PL, or its equal PF is to PD (4. 6 *Eu.*); and, because PF is equal to PL, the angle PFL is equal to PLF (5. 1 *Eu.*), or its equal (29. 1. *Eu.*) KMF; whence the angle KFM, being greater than PFM (*Ax.* 9. 1 *Eu.*), is also greater than KMF, and therefore KM is greater than KF (19. 1 *Eu.*), and of course the ratio of KF to KH is less than that of KM to KH (8. 5 *Eu.*), or, which has been just proved equal, of PF to PD, or the determining ratio; therefore the point K is within the section (8. 1 *Sup*).

In like manner it may be proved, that any point whatever in the right line OP is within the section; therefore the same right line OP is wholly within the section.

Cor. 1. If two right lines be cut by three or more parallel right lines, they are cut proportionally; it having been demonstrated in this proposition, from the parallelism of GO, HK and DP, that GH is to HD, as OK to KP; and a like reasoning being applicable to a greater number of parallel right lines.

Cor. 2. If two right lines (CD, FL) be cut proportionally by three right lines (GF, HM and DL), two of which are parallel to each other, the third is parallel to the other two: it having been demonstrated in this proposition, from GD and FL being cut proportionally, and GF being parallel to DL, that HM is parallel to both.

Cor. 3. If two right lines meeting each other, be parallel to two others meeting each other, the angle made by the former, is equal to that made by the latter towards the same part: it having been demonstrated in this proposition, from the parallelism of GO and OF to DP and PL, that the angle GOF is equal to DPL.

Scholium. In this 3d cor. it is required, that the angles be towards the same part, for if DP and LP were produced beyond P, they would form four angles about that point, whereof DPL and its opposite would be equal to GOF, and either of the other two, its complement to two right angles, as is manifest from 13. 1 Eu.

PROP. X. THEOR.

A right line, passing through any point of a conick section, and bisecting, in the case of an ellipse, an angle, in continuation to that, formed by right lines, drawn from that point to the focuses; and, in the case of a hyperbola, the angle so formed by right lines drawn to the focuses, touches the section, in that point only; as does, in the case of a parabola, a right line, passing through any point of the section, and bisecting the angle formed by two right lines, drawn from that point, one to the focus, and the other perpendicularly to the directrix.

Let GPK be a right line, passing through any point P of a conick section, see fig. 1, 2 and 3, and bisecting, in the case of an ellipse, fig. 1, the angle FPH, which is in continuation to that EPF, formed by right lines PE and PF drawn from P to the focuses E and F; in the case of a hyperbola, fig. 2, the angle EPF, so formed by right lines PE and PF drawn to the focuses; and, in the case of a parabola, fig. 3, the angle

HPF, formed by right lines PF and PH, drawn from P, to the focus F, and perpendicularly to the directrix DH; the right line PK is in each of the cases, a tangent to the section.

Case 1. When the figure is an ellipse (see fig. 1).

Fig. 1.

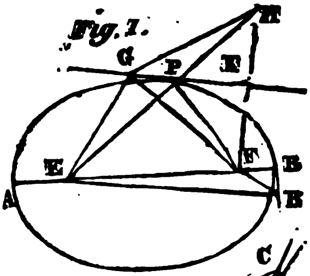


Fig. 2.

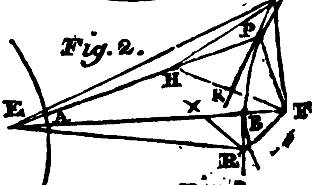
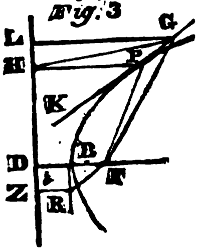


Fig. 3



Take any point whatever in GPK, except P, as G, and on EP produced, take PH equal to PF, and join GE, GF and GH; in the triangles GPF and GPH, the sides GP and PF are severally equal to GP and PH, and the included angles GPF and GPH are equal, being the same as the angles FPK and HPK, which are equal by hyp. when G is taken within the angle FPH, and, when taken otherwise, the complements of these equal angles to two right angles (13. 1 *Eu.*), therefore GF is equal to GH (4. 1 *Eu*). But in the triangle EGH, the sides EG and GH or its equal GF, are together greater than EH (20. 1 *Eu.*), or, which is equal (*Constr. and Ax.* 2. 1 *Eu.*), than EP and PF together; therefore the point G is without the ellipse (4. 1 *Sup*). In like manner any point whatever in GPK, except P, may be proved to be without the ellipse; and therefore PK, meeting the ellipse in the point P, and, if produced both ways, falling wholly without it, touches it in that point only (*Def.* 10. 1 *Sup*).

Case 2. When the figure is a hyperbola, see fig. 2.

Take any point whatever in GPK, except P, as G, on PE, take PH equal to PF, and join GE, GF and GH; and, in like manner as in the preceding case, GF and GH may be proved equal. And since, in the triangle EGH, EG is less than EH and HG together (20. 1 *Eu.*), taking from each GH, the excess of EG above GH or GF, is less than EH (*Ax.* 5. 1 *Eu.*), or, than the excess of FP above PH, or PF or (1. 1 *Sup.*), than AB; therefore the point G is without the hyperbola (4. 1 *Sup.*), whence it follows, as in the preceding case, that PK meeting the hyperbola in P, and, if produced both ways, falling wholly without it, touches it in that point only (Def. 10. 1. *Sup*).

Case 3. When the figure is a parabola, see fig. 3.

Take any point whatever in GPK, except P, as G, draw GL at right angles to DH, and join GF and GH; and, in like manner as in the first case, GF and GH may be proved equal; and, in the triangle GLH, because the angle GLH is a right angle, the angle GHL is acute (32. 1 *Eu.*), therefore GH, or its equal GF, is greater than GL (19. 1 *Eu.*), and so the point G is without the parabola (4. 1 *Sup.*), and therefore, as in the two preceding cases, PK, meeting the section in P, and, if produced both ways, falling wholly without it, touches it in that point only (*Def.* 10. 1 *Sup*).

Scholium. In like manner it may be proved, that a right line BR, see fig. 1, 2 and 3, perpendicularly meeting the principal axis BF of a conick section, in its vertex, touches the section in that point. Take any point whatever in BR, except B, as R; and, in fig. 1, drawing ER and FR, in the right angled triangles EBR and FBR, ER is greater than EB (*Cor.* 19. 1 *Eu.*), and FR than FB (by the same), therefore ER and FR together, are greater than EB and FB together, or AB, and so R is without the section (4. 1 *Sup.*): in the case of fig. 2, having taken on BE, a part BX equal to BF, and drawn ER, FR and XR, the right lines XR and FR are equal (4. 1 *Eu.*); whence, ER being less than EX and XR together (20. 1. *Eu.*), taking from each XR, the excess of ER above XR or above FR, is less than EX (*Ax.* 5. 1 *Eu.*), or than the excess of EB above XB, BF or EA, or, (1. 1. *Sup.*), than AB, and so R is again without the section (4. 1 *Sup.*); and, in the case of fig. 3, having drawn RF, and RZ perpendicularly to the directrix, RF is greater than BF (*Cor.* 1. 19. 1 *Eu.*), or its equal (*Def.* 8. 1. *Sup.*) DB, or, which is equal (34. 1 *Eu.*) ZR, and so R is, in this case also, without the section (4. 1 *Sup*). Since therefore, in every case, the right line BR meets the section in B, and, being produced both ways, falls wholly without it, BR touches the section in that point only (*Def.* 10. 1 *Sup*).

The case of a right line, perpendicularly meeting the second axis of an ellipse, in its vertex, comes under the first case of this proposition; since, in that case, the right line, so meeting the second axis, forms equal angles with right lines drawn from its vertex to the focuses, and, of course, bisects the angle in continuation of that formed by them, as might be easily shewn from the 1st figure to 1. 1 Sup. where the angles EMC, and FMC, which are the complements of the others to right angles, are equal (4. 1 *Eu*).

Cor. 1. From this proposition and scholium, it appears, how, the focus and principal vertex of a parabola, or the focuses of an ellipse or hyperbola being given, a right line may be drawn touching the section in a given point. Namely, by drawing from the given point in the case of a parabola, two right lines, one to the focus, and the other perpendicularly to the directrix; and, in the other cases, right lines to the focuses; the right line, bisecting the angle, formed by the right lines so drawn, in the cases of a hyperbola or parabola, and the angle in continuation thereto, in the case of an ellipse, is a tangent to the section (10. 1 *Sup*). But if the given point be the vertex of an axis, a right line, drawn through the given point, perpendicular to the axis, is the tangent required (*Schol.* 10. 1 *Sup*).

And since these focuses and principal vertices, are necessary parts of these figures, as is manifest from the definition of them; it follows, that a right line may touch a conick section in any point thereof.

Cor. 2. It also appears from this proposition, that, as a perpendicular is the nearest distance of a point from a right line (*Cor.* 1. 19. 1 *Eu.*); so the sum of two right lines (EP and PF, see fig. 1 of this prop.), drawn from two points (E and F) on the same side of a right line (GK), to any point in it (as P), is least, when the right line (GK), to which they are drawn, makes equal angles (EPG and FPK) with them, towards opposite parts (G and K).

For it is demonstrated in this proposition, that the sum of any other two, as EG and GF, so drawn, is greater than the sum of these.

Cor. 3. And hence appears a method of drawing to a given right line (GK), from two points (E and F) on the same side of it, right lines, to make with it equal angles, at the same point, towards opposite parts. Namely, by letting fall the perpendicular FK from one of the given points F, on GK, taking, on FK produced, KH equal to FK, and joining EH, which would cut GK in the point P required.

For the angle FPK is equal to HPK (4. 1 *Eu.*), or its equal (15. 1 *Eu.*) EPG.

Cor. 4. The differences of the distances, of two right lines (EP and PF see fig. 2), drawn from two points (E and F), on different sides of, and at unequal distances from, a right line (GR), to any point (P) in it, is greatest, when the right line (GK), to which they are drawn, makes equal angles (EPK and FPK) with them, towards the same part.

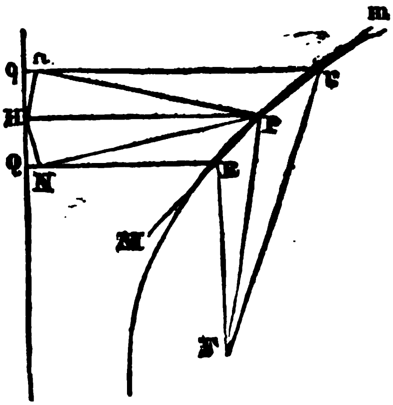
For it is demonstrated in this proposition, that the difference of any other two, as EG and GF, so drawn, is less than the difference of these.

Cor. 5. And hence appears a method of drawing to a given right line (GK), from two points (E and F), on different sides of, and at unequal distances from it, right lines, to make with it equal angles, at the same point, and towards the same part. Namely, by letting fall the perpendicular FK, from one of the given points F, on GK; taking on FK produced, KH equal to FK, and joining EH; which, because the points E and F, or E and H, are at unequal distances from GK, being produced towards the least distance HK, would meet GK, let it, so produced, meet it in P, and join FP.

The angles HPK or EPK and FPK are equal (4. 1 *Eu*).

Cor. 6. No right line can be drawn, in any conick section (see fig. 1, 2 and 3), between the right line GPK, bisecting the angle FPH, and the section, and of course any right line, drawn through P, which makes unequal angles with the right lines PF and PH, enters the section on the part of the point P, whether towards G or K, on which the angle formed with PF is less than that formed with PH.

For such a right line would divide the angle FPH unequally; whence, in the case of the ellipse and hyperbola, see fig. 1 and 2, right lines being drawn from E and F, to make equal angles with that right line at the same point, towards opposite parts in fig. 1, and to the same part in fig. 2 (by cor. 3 and 5 above), the sum of these, in the former case, would be less than of EP and PF (*Cor.* 2 *above*), and the difference in the latter case, greater than of EP and PF (*cor.* 4 *above*); therefore, in both cases, the point, to which the right lines are so drawn from E and F, is within the section (4. 1 *Sup.*); therefore the right line, so dividing the angle FPH unequally, enters the section; and, that it enters it on that side of P, on which the angle it makes with PF is less, is manifest, since, on the other side of P, it falls without GPK with respect to the section, and of course wholly without the section.



And, in the case of a parabola, see adjacent fig. a right line PM, which makes a less angle with PF than with PH, enters the section on the side of P which is towards F; for, from the greater angle MPH take MPN equal to MPF, make PN equal to PH, join NH, through N draw QNR at right angles to the directrix QH, meeting PM in R, and join RF.

Because PN is equal to PH, the triangle PHN is isosceles, therefore the angle PHN, being equal to PNH (5. 1 *Eu.*), is less than the right angle PHQ (32. 1 *Eu.*), therefore the point N falls between Q and R, and because RP, PF and the included angle RPF, are severally equal to RP, PN and the included angle RPN, RF is equal to RN (4. 1 *Eu.*), and therefore less than QR, and so the point R is within the section (4. 1 *Sup*).

And if a right line Pm make, on the other side of P, a less angle with PF than with PH; by using a similar construction and demonstration, as in the preceding case, only substituting the small letters m, n, q, r for the corresponding capitals, it may be shewn, that the point r in Pm is within the section; so that, in the case of a parabola, as well as of the other sections, a right line passing through P, and making unequal angles with PF and PH, enters the section on the part of P, on which the angle formed with PF, is less than that formed with PH.

Cor. 7. Through any point of a conick section, there can be drawn but one right line, touching it in that point. It having been proved in the preceding corollary, that any other right line, except that bisecting the angle FPH, enters the section, and is not of course a tangent (Def. 10 of this).

The condition in the second corollary to this proposition, towards opposite parts, is quite necessary; for if the points E and F, see fig. 1, be at unequal distances from GK, the right line EF, being produced toward the least distance, would meet GK, and form the same, and of course, equal angles with GK, but to the same part; a like observation is applicable to the condition in the fourth corollary. towards the same part; for the points E and F, see fig. 2, being on different sides of GK, the right line EF would intersect GK produced, and form with it equal angles, but towards opposite parts (15. 1 *Eu*).

PROP. XI. THEOR.

A tangent (GPK, see fig. 1, 2 and 3 of prec. prop.) to a conick section, bisects, in ellipses, the angle (FPH, see fig. 1), in continuation to that (EPF), formed by right lines drawn from the contact (P) to the focuses; in hyperbolas, that (EPF, see fig. 2), formed by right lines so drawn to the focuses; and, in parabolas, that (FPH, see fig. 3), formed by two right lines drawn from the contact (P), one (PF) to the focus, and the other (PH), perpendicularly to the directrix (DH).

For in every case, if the tangent PK bisects not the angle FPH, let there be drawn a right line bisecting this angle (9. 1 *Eu.*), the right line so drawn also touches the section in P (10 1. *Sup.*); which cannot be (*Cor.* 7. 10. 1 *Sup.*), therefore the tangent GPK bisects in every case the angle FPH.

Cor. 1. A right line, touching a conick section in the vertex of an axis, is perpendicular to the axis; for if not, a perpendicular to the axis being drawn at the vertex, would be a tangent to the section (*Schol.* 10. 1 *Sup.*), and so two right lines would touch the section in the same point, contrary to cor. 7 10. 1 Sup.

Cor. 2. A right line, drawn from any point of a conick section, perpendicularly to the axis, is ordinately applied to it; being parallel to the tangent, passing through a vertex of the axis, which is perpendicular to the axis (*prec. cor.*); the part between the section and axis, being an ordinate to the axis (*Def.* 12. 1 *Sup*).

Cor. 3. An ordinate to an axis, is perpendicular to it, because the tangent of the vertex, to which the ordinate is parallel (*Def.* 12. 1 *Sup.*), is perpendicular to the axis (*cor.* 1 *above*).

Cor. 4. If a right line, joining the vertices of two diameters of a parabola, be ordinately applied to the axis, their parameters are equal.

For the segments of these diameters, between their vertices and the directrix, are equal, being opposite sides of a parallelogram, and therefore the parameters, being fourfold of these segments (Def. 16. 1 *Sup*).

PROP. XII. THEOR.

A right line, passing through a focus of a conick section, and any point in the adjacent directrix, is perpendicular to a right line, joining that focus, to the contact, of a tangent to the section, drawn from the same point in the directrix; and makes equal angles, with two right lines, joining that focus, to the intersections, with the section or opposite sections, of any right line, drawn from the same point in the directrix, and cutting in two points the section or sections; the equal angles being on the same or different sides, of the right line joining the directrix and focus, and towards opposite parts or the same, according as the intersections of the secant, are in the same or opposite sections.

Let DFL, see fig. 1, 2 and 3, be a right line passing through a focus F, of a conick section, and a point D, in the directrix DH, adjacent to that focus, and let DR be a tangent to the section, drawn from D, and touching it in R ; and DQP a right line, drawn from D, cutting the section or opposite sections in P and Q, and let FR, FP and FQ be drawn : DL is perpendicular to FR, and makes equal angles PFG and QFD with FP and FQ ; the equal angles PFG and QFD being on the same or different sides of DFL, and towards opposite parts or the same, according as the intersections P and Q are in the same or opposite sections.

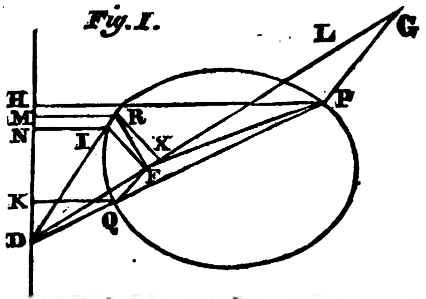


Fig. I.

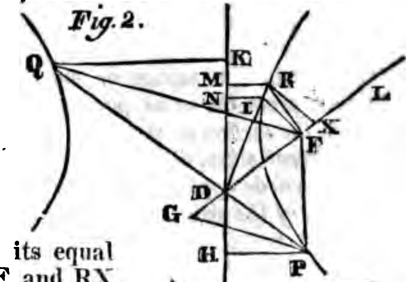


Fig. 2.

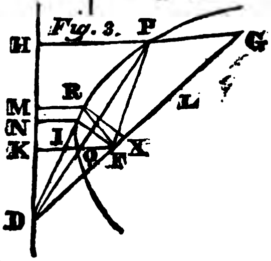


Fig. 3.

Part 1. The right line DL is perpendicular to the right line FR drawn to the tangent.

For, if the angles DFR and LFR be not right angles, and of course equal to each other, let one of them, as DFR, be, if possible, the greater, and take from it, a part DFI, equal to LFR, and let FI meet DR in I, let fall the perpendiculars RM and IN on DH, and through R, draw RX parallel to FI, meeting DL in X.

Because of the equiangular triangles DFI, DXR, and DIN, DRM, the angle RXF is equal to IFD or its equal *(constr.)* RFX, therefore RF and RX are equal (5. 1 *Eu.*) ; and FI is to ID, as XR, or its equal FR, is to RD (4. 6 *Eu.*), and ID to IN, as RD to RM (by the same), therefore, by equality, FI is to IN, as FR is to RM (22. 5 *Eu.*), or in the determining ratio, therefore the point I is in the section (8. 1 *Sup.*), and so the tangent DR meets the section in more than one point, which is absurd (10. 1 *Sup.*); therefore the angles DFR and LFR are not unequal, they are therefore equal, and of course DL is perpendicular to FR (*Def.* 20. 1 *Eu*).

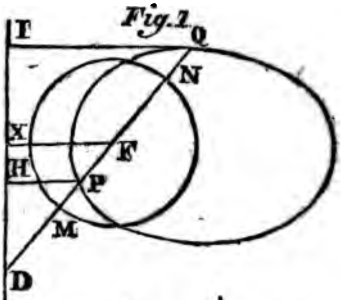
Part 2. The right line DL makes equal angles, with the right lines FP and FQ, drawn from the focus, to the points, wherein the secant DQP meets the section or opposite sections; the equal angles being on the same or different sides of DL, and towards opposite parts or the same, according as the intersections P and Q are in the same or opposite sections.

Through P, draw PG parallel to FQ, meeting DL in G; and, because of the equiangular triangles DPG, DQF, and DPH, DQK, GP is to PD, as FQ is to QD (4. 6. *Eu.*), and PD to PH, as QD to QK (by the same), therefore, by equality, GP is to PH, as FQ to QK (22. 5 *Eu.*), or, which is equal (6. 1 *Sup.*), as FP is to PH; whence GP and FP, having the same ratio to PH, are equal (9. 5. *Eu*), therefore the angle PFG is equal to the angle PGF, or, which is equal (29. 1 *Eu.*), the angle QFD, and so the right line DL makes equal angles with the right lines FP and FQ, on the same side of DL, and towards opposite parts, when P and Q are in the same section, as in fig. 1 and 3, but on different sides of DL, and towards the same part, when P and Q are in the opposite sections, as in fig. 2.

PROP. XIII. THEOR.

If, in a right line, touching, or cutting in two points, a conick section or opposite sections, and meeting a directrix, any point be taken, and also a finite right line, which is to the distance of that point from the directrix, in the determining ratio; the square of the segment of the tangent, or rectangle under the segments of the secant, between the assumed point, and the point or points, wherein it meets the section or sections, is to the difference of the squares, of the distance of that point, from the focus adjacent to the directrix, and the assumed finite right line; as the square of the segment, of the same tangent or secant, between the same point and the directrix, is to the difference of the squares, of the same segment, and the assumed finite right line.

Fig. 1.



Case 1. When a secant DPQ passes through a focus F, that focus being the assumed point.

And P and Q being the points, in which the secant meets the section, and D that, in which it meets the directrix; the perpendicular FX being let fall from F on the directrix DX, and FM being taken on FD, having to FX the determining ratio, see schol. 6. 1 Sup.—The rectangle PFQ is to the square of FM, as the square of FD is to the difference of the squares of FD and FM.

From P and Q let fall the perpendiculars PH and QI on the directrix DX, and from the centre F, at the distance FM, describe a circle, meeting DPQ again in N.

And, since, as FP is to PH, so is FM to FX [Hyp.], and, because of the equiangular triangles PHD and FXD, PH is to PD, as FX is to FD (4. 6 *Eu.*), by equality, FP is to PD, as FM is to FD (22. 5 *Eu.*), and by con-

Fig. 2.

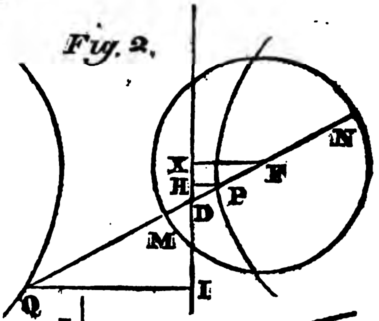
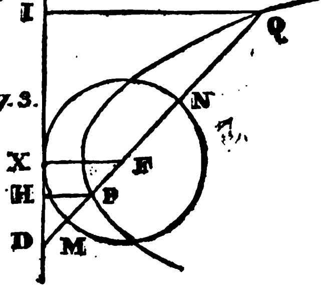


Fig. 3.



verting, FP is to the sum of FP and PD, or FD, as FM is to the sum of FM and FD, or ND (*Schol.* 18. 5. *Eu.*], and, alternating, FP is to FM, as FD is to ND.

In like manner, it may be proved, that FQ is to FM, as FD is to MD ; by using the reasoning of the preceding paragraph, only substituting the letters Q, I and N for the letters P, H and M respectively, and the word "difference" for the word "sum"; therefore, compounding this ratio and that of the same paragraph, the rectangle PFQ is to the square of FM, as the square of FD is to the rectangle MDN (23. 6 and 22. 5 *Eu*), or, which is equal (*Cor.* 3. 36. 3 *Eu*), to the difference of the squares of FD and FM.

Fig. 1.

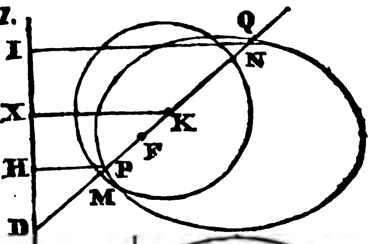


Fig. 2.

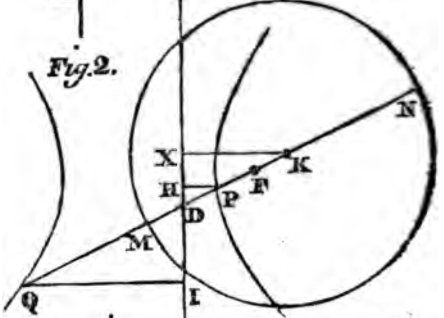
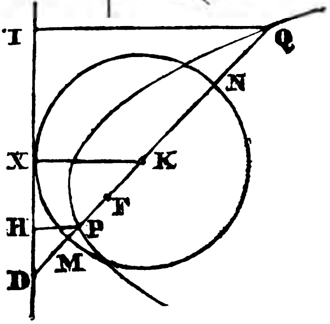


Fig. 3.



Case 2. When a secant DPQ passes thro' a focus F, the assumed point not being the focus, but some other point in the secant, as K.

The points, wherein the secant meets the section and directrix, being noted as before, let fall the perpendicular KX on the directrix, and, taking KM on KD having to KX the determing ratio.—The rectangle PKQ is to the difference of the squares of KF and KM, as the square of KD is to the difference of the squares of KD and KM.

Having let fall the perpendiculars PH and QI on the directrix DX, from the centre K, at the distance KM, describe a circle meeting DPQ again in N.

Because of the equiangular triangles FHD and KXD, PD is to PH, as KD is to KX (4. 6. *Eu.*), and as PH is to PF, so is KX to KM (*Hyp. and Theor.* 3. 15. 5 *Eu.*), therefore by equality, PD is to PF, as KD is to KM (22. 5 *Eu.*), and, by alternating, PD is to KD as PF is to KM (16. 5 *Eu.*), therefore the excess of KD above PD, or KP, is to KD, as the excess of KM above PF, is to KM (17. 5 *Eu.*), and therefore, alternating, KP is to the excess of KM above PF, as KD is to KM (16. 5 *Eu.*), and converting, KP is to KP with the excess of KM or KN above PF, or FN, as KD is to KD and KM together, or ND.

In like manner, because of the equiangular triangles DXK and DIQ, KD is to KX, as QD to QI (4. 6 *Eu.*), and as KX is to KM, so is QI to QF (*Hyp. and Theor.* 3. 15. 5 *Eu.*), therefore, by equality KD is to KM, as QD to FQ (22. 5 *Eu.*), and, alternating, KD is to QD, as KM to FQ (16. 5. *Eu.*), therefore the sum of KD and QD in fig. 2, and their difference in the other figures, or KQ, is to KD, as the sum or difference, as the case may be, of KM and FQ, is to KM (17 and 18. 5 *Eu.*), and, alternating, KQ is the sum or difference, as the case may be, of KM and FQ, as KD is to KM (16. 5 *Eu.*), and therefore, converting, KQ is to the difference of KQ and the sum or difference of KM and FQ, which is, in every case, equal to FM, as KD is to the difference of KD and KM, or MD (*Schol.* 18. 5 *Eu*). Therefore, compounding this ratio and that of the preceding paragraph, the rectangle PKQ is to the rectangle MFN, or, which is equal (*Cor.* 3. 36. 3 *Eu.*), the difference of the squares of KF and KM, as the square of KD is to the rectangle MDN, or, which is equal (by the same), the difference of the squares of KD and KM (23. 6 and 22. 5 *Eu*).

Case 3. When a secant DPQ, meeting a directrix DX, does not pass through the adjacent focus.

Let the assumed point be K, join FP, FQ, FK and DF, which last produce as to N, from the assumed point K draw KM and KN parallel to FQ and FP, meeting DFN in M and N, and draw KX and PH at right angles to DX: and, because of the parallels KN and PF, the angle KNM is equal to PFD (29. 1 *Eu.*), or, which is equal (12. 1 *Sup.*), the angle QFN in fig. 1 and 3, and QFD in fig. 2, or, which, because of the parallels QF and KM, is equal (29. 1 *Eu.*), KMN; therefore KM and KN are equal (6. 1 *Eu.*), and a circle described from the centre K at the distance KM passes through N; let this circle be described, and because of the equiangular triangles NKD, FPD, and KDX, PDH, NK is to KD, as PF is to PD, and KD to KX, as PD to PH (4. 6. *Eu.*), therefore, by equality, NK or MK is to KX, as FP to PH (22. 5 *Eu.*), or in the determining ratio.

And the rectangle PKQ is to the difference of the squares of KF and KM, as the square of KD is to the difference of the squares of KD and KM.

Fig. 1.

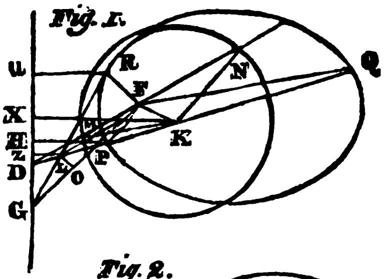


Fig. 2.

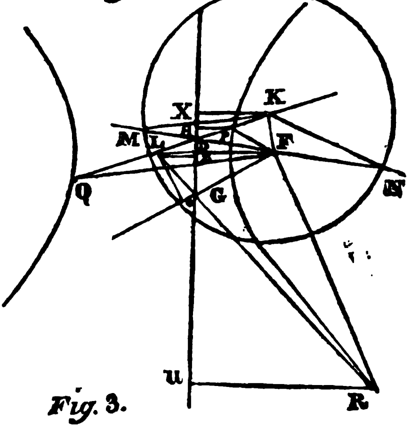
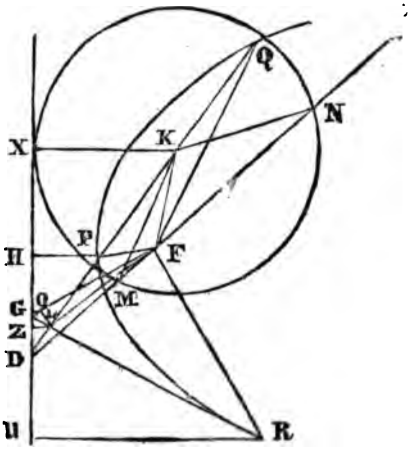


Fig. 3.



For, because KN is parallel to FP, and MK to FQ; PK is to FN, as KD is to ND (2. 6 and 18 and 16. 5 *Eu.*), and QK to FM, as KD to MD (by the same); therefore, compounding these two ratios, the rectangle PKQ is to the rectangle MFN, or, which is equal (*Cor.* 3. 36. 3), the difference of the squares of KF and KM, as the square of KD is to the rectangle MDN, or, which is equal (by the same), the difference of the squares of KD and KM (23. 6 *and* 22. 5 *Eu*).

Case 4. When the assumed point as L, is in a tangent GLR meeting a directrix DX in G.

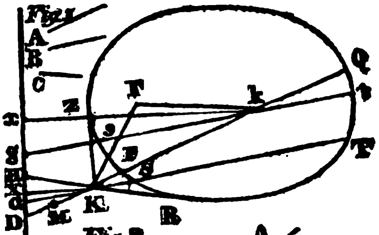
From the focus F adjacent to the directrix DX, to the contact R, draw FR, join GF, draw RU at right angles to DX, and LO parallel to RF, meeting FG in O. And, because the angle GFR is a right angle (12. 1 *Sup.*), the angle GOL, equal to it (29. 1), is also a right angle; and, because of the equiangular triangles GOL and GFR, GLZ and GRU, LO is to LG, as RF is to RG (4. 6 *Eu.*), and LG to LZ, as RG to RU (by the same), therefore, by equality, LO is to LZ, as RF to RU (22. 5 *Eu.*), or in the determining ratio.

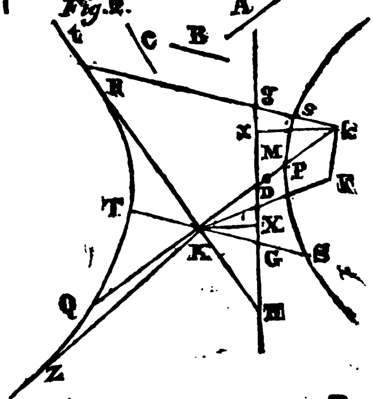
And the square of LR is to the difference of the squares of LF and LO, as the square of LG is to the difference of the squares of LG and LO.

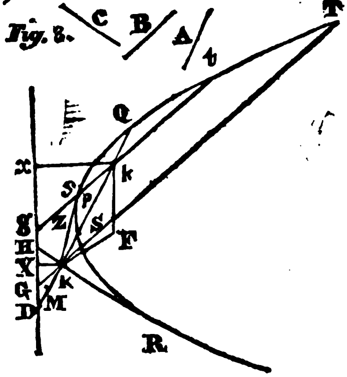
For, because of the parallels OL and FR, the square of LR is to the square of OF, or, which is equal (47. 1 *Eu.*), the difference of the squares of LF and LO, as the square of LG is to the square of GO (2 *and* 22. 6 *and* 16. 5 *Eu.*), or, which is equal (47. 1 *Eu.*), to the difference of the squares of LG and LO.

PROP. XIV. THEOR.

If two right lines, meeting each other, and parallel to two right lines given by position, both touch, or both cut in two points, or one of them touch and the other so cut a conick section, or opposite sections; the squares of the segments of the tangents, or rectangles under the segments of the secants, between the concourse of the right lines, and the section or sections, are to each other, always in the same ratio, wherever the concourse of the right lines may fall.







Case 1. Let the right lines, parallel to the right lines A and B given by position, be the secants DPQ and GST meeting each other in K, and a directrix DX in D and G.—The ratio of the rectangle PKQ to SKT is a constant ratio, and the same, wherever the point K may fall.

Let KF be drawn to the focus F adjacent to the directrix DX, and the perpendicular KX be let fall on the directrix DX, and let KM be taken, having to KX the determining ratio; then the rectangle PKQ is to the difference of the squares of KF and KM, as the square of KD is to the difference of the squares of KD and KM (13. 1 *Sup.*); also the rectangle SKT is to the difference of the squares of KF and KM, as the square of KG is to the difference of the squares of KG and KM (by the same), and, by inverting, the difference of the squares of KF and KM is to the rectangle SKT, as the difference of the squares of KG and KM is to the square of KG (*Theor.* 3. 15. 5 *Eu.*); but the rectangle PKQ is to the rectangle SKT in a ratio compounded of the ratios of the rectangle PKQ to the difference of the squares of KF and KM, and of the difference of the same squares of KF and KM to the rectangle SKT [*Def.* 13. 5 *Eu.*], or, which has been just shewn to be equal, ratios compounded of equal ratios being equal by 22. 5. Eu. in a ratio compounded of the ratios of the square of KD to the difference of the squares of KD and KM, and of the difference of the squares of KG and KM to the square of KG.

But the position of A, and of KPQ parallel to it, as also of the directrix DX, being given, the angle KDX is given, whence, the angle KXD being right, and therefore given, the ratio of KD to KX is a constant ratio, or, the same, wherever in the right line DQ the point K may be (4. 6 *Eu.*), also the ratio of KX to KM, being the determining ratio, is a constant ratio (6. 1 *Sup.*), therefore the ratio of KD to KM, compounded of both these ratios (Def. 13. 5 *Eu.*), is a constant ratio (22. 5 *Eu.*), and therefore also the ratio of the square of KD to the square of KM (20. 6 *and Cor.* 3. 22. 5 *Eu.*), and therefore of the square of KD to the difference of the squares of KD and KM (*Schol.* 18. 5 *Eu.*); in like manner, and by inverting, it may be proved, that the ratio of the difference of the squares of KG and KM to the square of KG is constant; therefore the ratio, which is the compound, of the ratio of square of KD to the difference of the squares of KD and KM, and of that of the difference of the squares of KG and KM to the square of KG, is constant, (22. 5 *Eu.*): but the rectangle PKQ has been just shewn to be to the rectangle SKT, in a ratio compounded of these ratios, therefore these rectangles are to each other in a constant ratio, and therefore in the same ratio, wherever the concourse K of the right lines KPQ and KST may fall.

Case 2. Let now one of the right lines, as HKR, parallel to a right line as C, given by position, and meeting a directrix, as in H, be a tangent, the point of contact being R, the other DPQ being a secant as in the former case, the point of concourse of these right lines being K. The ratio of the rectangle PKQ to the square of KR is a constant ratio, and the same, wherever the point K may fall.

The demonstration is exactly the same as that of the preceding case, only substituting for the rectangle SKT, the square of KR, and for the right lines KG and KST, the right lines KH and KR.

Case 3. Again, let both the right lines, so meeting the directrix and each other, be tangents, as KZ and KR; their concourse being K, and the points of contact Z and R.

Through their concourse K, draw any right line KPQ, cutting the section or opposite sections in P and Q. By the preceding case, the square of KZ is to the rectangle PKQ in a constant ratio, and, by the same, the rectangle PKQ is to the square of KR in a constant ratio, therefore, by equality, the squares of KZ and KR are to each other in a constant ratio.

Fig.1

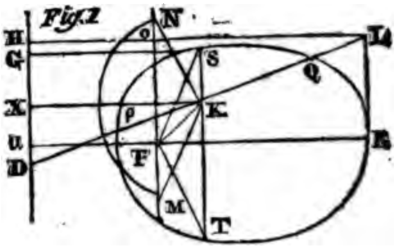


Fig.2

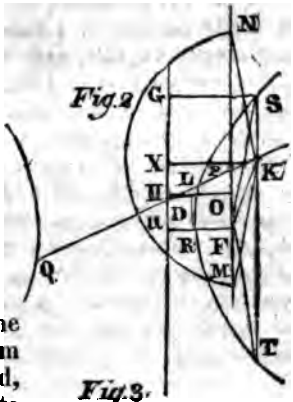
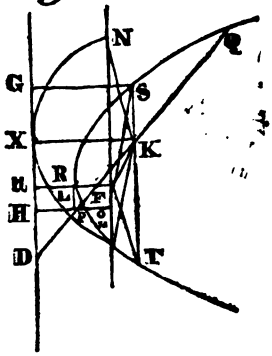


Fig.3.



Case 4. Let now one of two secants PQ and ST, see fig. 1, 2 and 3, meeting each other in K, and the section or sections in P and Q, S and T, as ST, be parallel to the directrix DX, PQ meeting the directrix in D.—The rectangle PKQ is to the rectangle SKT in a constant ratio.

Join FS and FT, through the focus F, let MFN be drawn, parallel to DG, and from K draw KM and KN parallel to FS and FT, meeting MFN in M and N, let fall the perpendiculars KX and SG on DX, and join FK.

Because ST is parallel to DG, the distances of the points S, K and T from DG are equal (28 and 34. 1 *Eu.*); and, because ST is parallel to MN, KM to FS, and KN to FT, in the parallelograms MS and NT, the side MK is equal to FS and NK to FT, also FM to SK and NF to KT (34. 1 *Eu.*); and, because FS and FT have the same ratio to the equal distances of the points S and T from the directrix DG (6. 1 *Sup.*), they are equal (9. 5 *Eu.*), as are therefore their equals KM and KN; whence, a circular arch described from the centre K at the distance KM passes through the point N; let such an arch MN be described, and KM is to KX, as FS is to SG, or in the determining ratio (6. 1 *Sup.*), and so the rectangle PKQ is to the difference of the squares of KF and KM, or, which is, because of the circular arch MN, equal (*Cor.* 3. 36. 3 *Eu.*), the rectangle MFN, or, SK and KT being severally equal to MF and FN, to the rectangle SKT, as the square of KD is to the difference of the squares of KD and KM (13. 1 *Sup.*), and therefore in a constant ratio.

Case 5. Lastly, let one of the right lines PQ and LR, meeting each other in L, as LR, be a tangent, touching the section in a vertex R of the principal axis FR, and of course perpendicular to that axis (*Cor.* 1. 11. 1 *Sup.*), and therefore parallel to the directrix DG, the other PQ being a secant, meeting the directrix in D, and the section or sections in P and Q. The rectangle PLQ is to the square of LR, in a constant ratio.

Let fall the perpendiculars FU and LH on the directrix DX, draw FO parallel to LR meeting LH in O ; and, because of the parallelograms HR and OR, the right lines HL, OL and OF are severally equal to UR, FR and LR (34. 1 *Eu.*) ; therefore OL is to LH. as FR to RU, and therefore in the determining ratio (6. 1 *Sup.*) ; whence, a right line being supposed to be drawn from F to L, the rectangle PLQ is to the difference of the squares of FL and LO, or, because of the right angled triangle FOL and parallelogram OR, to the square of OF (47. 1 *Eu.*), or, which is equal (34. 1 *Eu.*), LR, as the square of LD is to the difference of the squares of LD and LO (13. 1 *Sup.*), and therefore, as before, in a constant ratio.

Scholium.—That the truth of this proposition may appear more clearly, in the different positions of the point K, another secant skt is exhibited in the figures to the three first cases, meeting the secant DPQ, in a different situation, with respect to the section, as internally, instead of externally ; the reasoning in the demonstration of the proposition applying, by substituting the small letters s, k, t, g and x, for their respective capitals.

Cor. 1. If any right line (KZ, see fig. 1 of this prop.), touching a conick section, meet two parallel right lines (GT, gt), cutting the section or opposite sections ; the rectangles under the segments of the secants, between the tangent and the section or sections, are to each other, as the squares of the segments of the tangent, between the parallels and the contact.

For the ratios of the rectangles under these segments of the secants, to the squares of the respective segments of the tangent which they meet, being by this prop. equal, by alternating these rectangles are to each other, as the same squares.

Cor. 2. Or if any right line, touching a conick section, meet two parallel right lines, touching the section or opposite sections ; it may in the same manner be proved, that the squares of the segments of the parallel tangents between their contacts and the tangent which they meet, are to each other, as the squares of the segments of that tangent, between the parallels and its contact.

Cor. 3. Or if any right line, touching a conick section, meet two parallel right lines, whereof one is a tangent, and the other a secant; the square of the segment of the tangent, and rectangle under the segments of the secant, between the tangent which they meet and the section, are to each other, as the squares of the segments of that tangent between the parallels and its contact.

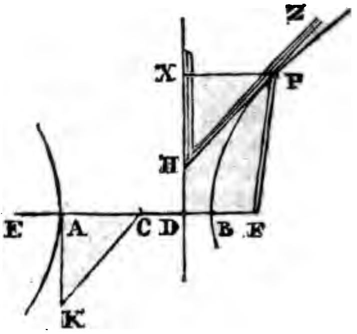
Cor. 4. Or if any right line, cutting a conick section or opposite sections in two points, meet two parallel right lines cutting in like manner the same section or sections; the rectangles under the segments of the parallels between the section or sections, and the right line which they meet, are to each other, as the rectangles under the segments of the right line which they so meet, between the parallels and the section or sections.

Cor. 5. Or if any right line, cutting a conick section or opposite sections, meet two parallel right lines touching the same section or opposite sections; the squares of the segments of the parallel tangents between their contacts, and the secant which they meet, are to each other, as the rectangles under the segments of that secant, between the parallels, and the section or sections.

Cor. 6. Or if any right line, cutting in two points a conick section or opposite sections, meet two parallel right lines, whereof one is a tangent, and the other a secant; the square of the segment of the tangent, and the rectangle under the segments of the secant, between the section or sections and the secant which they meet, are to each other, as the rectangles under the segments of that secant between the parallels, and the section or sections.

PROP. XV. THEOR.

A right line (PH), drawn from any point (P) of a hyperbola (BP), to the adjacent directrix (DX), parallel to the adjacent asymptote (CK), is equal to the distance (PF) of the same point, from the adjacent focus (F).



Let E be other focus, and AB the principal axis; draw AK at right angles to AB, meeting the asymptote CK in K, and PX at right angles to DX.

Because AK is equal to the second semiaxis (*Def.* 19. 1 *Sup.*), its square is equal to the difference of the squares of CE and CA (2. 1 *Sup.*), and the square of AK is also equal to the difference of the squares of CK and CA (47. 1 *Eu.*); whence, the difference of the squares of CE and CA, and of CK and CA, being each equal to the square of AK, are equal to each other (*Ax.* 1. 1), adding to each of these differences the square of CA, the squares of CE and CK are equal, and so CE is equal to CK; and PF is to PX, as CE, or, which has been just proved equal to it, CK, is to CA (*Schol.* 6. 1 *Sup.*), or, because of the equiangular triangles ACK and XHP, as PH is to PX (4. 6 *Eu.*); therefore PF and PH, having the same ratio to PX, are equal (9. 5 *Eu*).

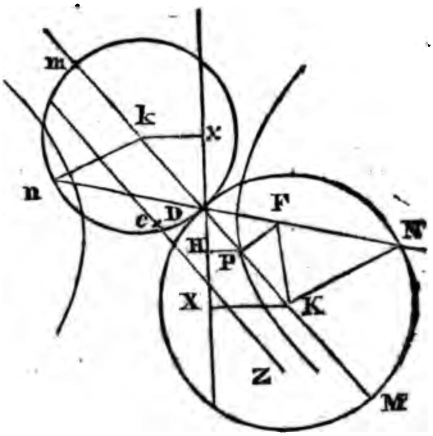
Scholium.—Hence, a focus F, the adjacent directrix DX, and an asymptote CK of a hyperbola being given, the section may be described. Let XHZ be an instrument similar to a square, but with one side HZ moveable about H, so as to make the angle ZHX equal to a given one. Let one side of it HX be applied to the directrix DX, and the other side HZ, being toward the part on which is F, be so inclined to the side HX, that it may be parallel to CK; and to the extremity Z, of the side HZ, let one extremity of a thread of the same length as HZ be fastened, and let its other extremity, the thread going round a pin in the side HZ, at the point P, be fastened at the point F, and because the thread FPZ is equal to HZ, taking PZ from each, PF remains equal to PH; let the side HX of the instrument be moved along the line DX, and, the thread remaining extended, let the pin, affixed to the side HZ of the instrument, describe the line BP, which is the hyperbola required, as is manifest from this proposition.

Hence appears further, the close analogy, which exists between the parabola and hyperbola, seeing that, if the side of the square, to which the thread and pin, are applied, deviate ever so little either way from a right angle with the other side, the figure becomes a hyperbola.

PROP. XVI. THEOR.

If, in a right line parallel to an asymptote of a hyperbola, any point be taken, and also a finite right line, which is to the distance of that point from the directrix, adjacent to the hyperbola which it meets, in the determining ratio; the rectangle under the distances of the point, wherein the parallel right line meets the hyperbola, from that wherein it meets the directrix, and from the assumed point, is to the difference of the squares of the distance of that point, from the focus adjacent to the same directrix, and the assumed finite right line, as the square of the segment of the same parallel, between the directrix and hyperbola, to the square of a right line, joining the focus, to the point, in which the parallel meets the directrix.

In a right line DK, parallel to an asymptote CZ of a hyperbola, let any point whatever K be taken; let P and D be the points in which DK meets the hyperbola and directrix DX adjacent thereto, and F the adjacent focus; let KM be taken on DH, having to a perpendicular KX, let fall from K on the directrix DX, the determining ratio, and let DF, PF and FK be joined.



The rectangle DPK is to the difference of the squares of KF and KM, as the square of PD is to the square of FD.

Let fall the perpendicular PH on DX, and from K draw KN parallel to PF, meeting DF, produced if necessary, in N.

Because of the equiangular triangles DKN and DPF, KN is to KD, as PF is to PD (4. 6 *Eu.*); whence, PF being equal to PD (15. 1 *Sup.*), KN is equal to KD. And since KN is to KD, as PF is to PD, and, because of the equiangular triangles DKX and DPH, DK is to KX, as DP to PH (4. 6 *Eu.*), by equality, KN is to KX, as PF to PH (22. 5 *Eu.*), or in the determining ratio; therefore KN or KD is equal to KM (*Hyp. and* 9. 5 *Eu*).

From the centre K, at the distance KD, KN or KM, let the circle DNM be described; and, because of the parallels PF and KN, the rectangle DPK is to the rectangle DFN, or, which is equal (*Cor.* 3. 36. 3 *Eu.*), the difference of the squares of KF and KM, as the square of DP is to the square of DF (20. 6 and *Cor.* 3. 22. 5 *Eu*).

Cor. The segment (KD), of a right line (DM) parallel to an asymptote (CZ) of a hyperbola, between any point (K) in the parallel, and the directrix, is to a perpendicular (KX) let fall from the same point on the directrix, in the determining ratio.

For, because of the equiangular triangles DKX and DPH, KD is to KX, as PD, or its equal (15. 1 *Sup.*) PF, is to PH (4. 6 *Eu.*), or in the determining ratio.

Scholium. The reasoning in this proposition and corollary applies, whether the point taken in the parallel, be within or without the hyperbola, as the point k, a right line being supposed to be drawn from k to F, by substituting the small letters k, x, m and n for their respective capitals.

PROP. XVII. THEOR.

If two right lines, parallel to each other, both touch or both cut in two points, or one of them touch, and the other so cut, a hyperbola or opposite hyperbolas, and meet a right line parallel to an asymptote; the squares of the segments of the tangents, or rectangles under the segments of the secants, between the right line parallel to the asymptote, and the point or points wherein they meet the hyperbola or hyperbolas, are to each other, as the segments of the right line parallel to the asymptote, between the parallels, and the concourse of that right line, with the hyperbola, which it meets.

Fig. 1.

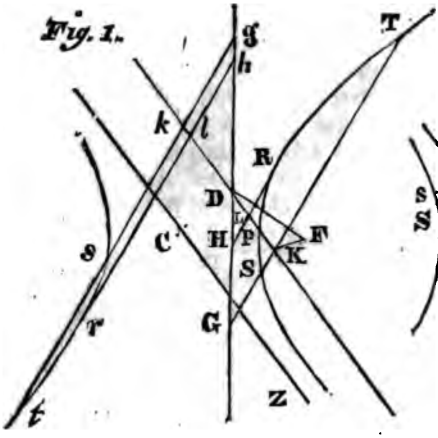
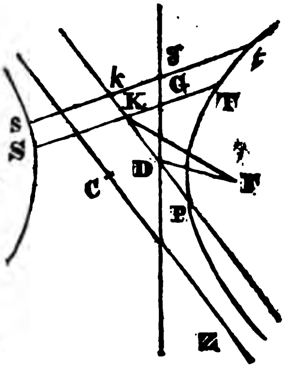


Fig. 2.



Let, in fig. 1, RL and rl be two tangents, parallel to each other, touching opposite hyperbolas in R and r, meeting a directrix DG, in H and h, and DK parallel to the asymptote CZ in L and l; and let secants ST and st meet a hyperbola, as in fig. 1, or opposite hyperbolas, as in fig. 2, in S and T, s and t, the directrix DG in G and g, and the parallel to the asymptote in K and k. The squares of LR and lr in fig. 1, and the rectangle SKT and skt in fig. 1 and 2, are to each other, as the right lines PL and Pl, PK and Pk.

Join KF and DF; and KD and kD in fig. 1 and 2, and LD and lD in fig. 1, have to perpendiculars let fall from the points K and k, L and l, on the directrix DG, the determining ratio (*Cor.* 16. 1 *Sup.*), and let, first, the parallel right lines which meet DK, be, in fig. 1, the tangent LR and the secant ST; and since the rectangle SKT is to the difference of the squares of KF and KD, as the square of KG is to the difference of the squares of KG and KD (13. 1 *and Cor.* 16. 1 *Sup.*), and the difference of the squares of KF and KD is to the rectangle DPK, as the square of DF is to the square of DP (16. 1 *Sup. and Theor.* 3. 15. 5. *Eu.*), by compounding these ratios, the rectangle SKT is to the rectangle DPK, in a ratio compounded of the ratios of the rectangle SKT to the difference of the squares of KF and KD, and of the same difference to the rectangle DPK, (*Def.* 13. 5 *Eu.*), or, which has been just shewn to be equal, of the ratios of the square of KG to the difference of the squares of KG and KD, and of the square of DF to the square of DP.

In like manner it may be proved, that the square of RL is to the rectangle DPL, in a ratio, compounded of the ratios of the square of LH to the difference of the squares of LH and LD, and of the square of DF to the square of DP.

But in these two compounded ratios, that of the square of DF to the square of DP is common to both, and, because of the equiangular triangles KGD and LHD, the ratios of KG to KD and of LH to LD are equal (4. 6. *Eu.*), and therefore those, of the square of KG to the square of KD, and of the square of LH to to the square of LD (20. 6. and *cor.* 3. 22. 5. *Eu.*), and therefore those, of the square of KG to the difference of the squares of KG and KD, and of the square LH to the difference of the squares of LH and LD (*Schol.* 18. 5 *Eu.*) therefore these compound ratios being compounded of equal ratios, are equal (22. 5 *Eu.*), therefore the ratios equal to them of the rectangle SKT to the rectangle DPK, and of the square of RL to the rectangle DPL are equal, and, by alternating, the rectangle SKT is to the square of RL, as the rectangle DPK, is to the rectangle DPL (16. 5 *Eu.*), or, the side DP being common to both rectangles, as PK is to PL (1. 6. *Eu*).

In like manner, if, instead of the tangent RL and secant SKT, the tangent rl and secant skt be used, the truth of the proposition may be shewn, a right line being supposed to be drawn from k to F, by substituting the small letters s, t, k, g, r, l and h for the corresponding capitals, as is manifest.

And, by a similar reasoning, it may be proved, in both figures, that the rectangles SKT and skt, are to each other, as the segments KP and kP, of the right line parallel to the asymptote, between the parallels ST and st, and the point P.

PROP. XVIII. THEOR.

If a right line, touching a hyperbola, or cutting a hyperbola or opposite hyperbolas in two points, meet two right lines parallel to an asymptote; the squares of the segments of the tangent, or rectangles under the segments of the secant, between the parallels, and the point or points, wherein the tangent or secant meets the section or sections, are to each other, in a ratio, compounded of the ratios, of the segments of the parallels, between the tangent or secant, and the points wherein the same parallels meet the hyperbola, and of the segments of the same parallels, between a right line passing through the focus adjacent to them, and perpendicularly cutting them, and the adjacent directrix.

Fig. 1.

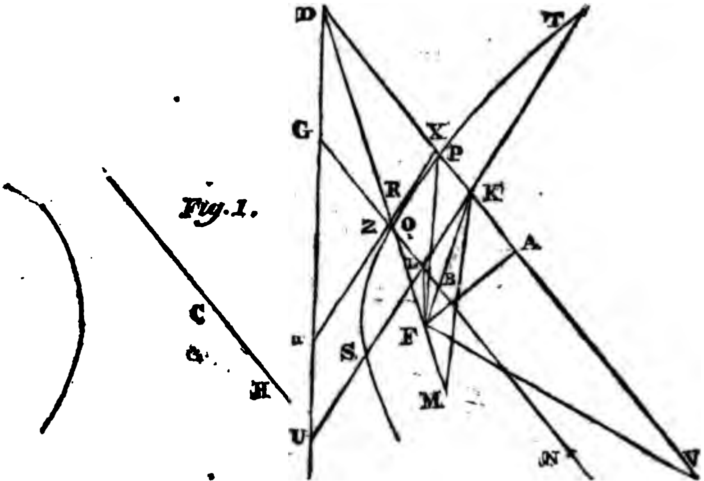
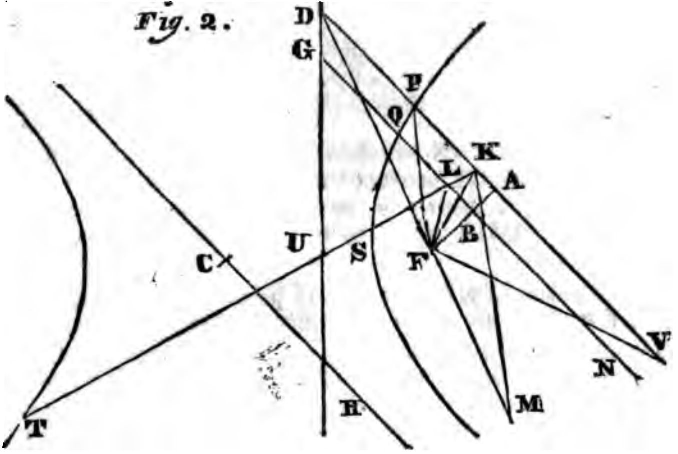


Fig. 2.



Let a right line XZ, see fig. 1, touching a hyperbola in R, or ST, see fig. 1 and 2, cutting a hyperbola or opposite hyperbolas in S and T, meet two right lines DK and GL, parallel to an asymptote CH, and meeting the adjacent directrix DG in D and G; from the focus F adjacent to the parallels, let the right line FBA be drawn, perpendicularly meeting these parallels in A and B, the squares of XR and ZR, or rectangles SKT and SLT, as the case may be, are to each other in a ratio compounded of the ratios of XP to ZQ or PK to QL, and of DA to GB.

First, the rectangles SKT and SLT, see fig. 1 and 2, are to each other. in a ratio compounded of the ratios of PK to QL and of DA to GB.

Join FD, FK, FL and FP; let U and u be the points in which ST and XZ meet the directrix DG; take AV equal to AD, and through K draw KM parallel to PF, meeting DF produced in M.

The right line FV is equal to FD (4. 1 *Eu.*), and PF to PD (15. 1 *Sup.*); therefore the triangles DFV and DPF are isosceles, and, having the angle PDF common, are equiangular, therefore DV is to DF, as DF is to DP (4. 6 *Eu.*), or, which is equal, because of the parallels PF and KM, as FM is to PK; (2. 6 and 16. 5 *Eu.*), therefore the rectangle under DV or twice DA and PK, is equal to the rectangle DFM (16. 6 *Eu.*): but, because of PF equal to PD, and parallel to KM, the right lines KD and KM are equal, and a circle described from the centre K, at the distance KD, would pass through M, and so the rectangle DFM is equal to the difference of the squares of KD and KF (*Cor.* 3. 36. 3 *Eu.*); therefore the rectangle under twice DA and PK is equal to the difference of the squares of KD and KF.

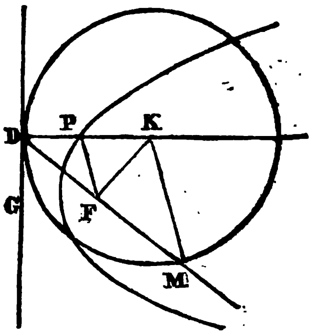
In like manner, if BN be taken equal to BG, and FQ, FG and FN be joined, and GF produced meet a right line drawn through L parallel to QF, it may be proved, that the rectangle under twice GB and QL is equal to the difference of the squares of LG and LF.

But the ratios of KD and LG, to perpendiculars let fall from K and L on DG, are equal to the determining ratio (*Cor.* 16. 1 *Sup.*), therefore the rectangle SKT is to the difference of the squares of KD and KF, as the square of KU is to the difference of the squares of KU and KD (13. 1 *Sup.*); for a like reason, the rectangle SLT is to the difference of the squares of LG and LF, as the square of LU is to the difference of the squares of LU and LG; but, because of the equiangular triangles KUD and LUG, the ratios of KU to KD and of LU to LG are equal (4. 6 *Eu.*), and therefore those, of the square of KU to the square of KD, and of the square of LU to the square of LG (20. 6 *and Cor.* 3. 22. 5 *Eu.*), and therefore those, of the square of KU to the difference of the squares of KU and KD, and of the square of LU to the difference of the squares of LU and LG (*Schol.* 18. 5 *Eu.*); therefore the ratios of the rectangle SKT to the difference of the squares of KD and KF, and of the rectangle SLT to the difference of the squares of LG and LF, which are equal to these equal ratios, are equal (11. 5 *Eu.*); and the difference of the squares of KD and KF is above proved equal to the rectangle under twice DA and PK, and the difference of the squares of LG and LF to the rectangle under twice GB and QL, therefore the rectangle SKT is to the rectangle under twice DA and PK, as the rectangle SLT is to the rectangle under twice GB and QL, and, by alternating, the rectangle SKT is to the rectangle SLT, as the rectangle under twice DA and PK is to the rectangle under twice GB and QL (16. 5 *Eu.*); or, which is equal (23. 6 *Eu.*), in a ratio compounded of the ratios of PK to QL, and of twice DA to twice GB; or, twice DA being to twice GB, as DA to GB (15. 5 *Eu.*), in a ratio compounded of the ratios of PK to QL and of DA to GB.

In like manner it may be proved, in the case of the tangent XZ, see fig. 1, by drawing through X and Z, instead of through K and L, right lines parallel to PF and QF, and otherwise constructing and reasoning as above, that the rectangles under twice DA and XP, and under twice GB and ZQ, are equal to the differences of the squares of XD and XF, and of ZG and ZF; and so the square of RX being, for the like reason as above, to the difference of the squares of XD and XF, as the square of RZ is to the difference of the squares of ZG and ZF; by substituting for the differences of the squares of XD and XF, and of ZG and GF, their equals as above, the rectangles under twice DA and XP, and under twice GB and ZQ, the square of RX is to the rectangle under twice DA and XP, as the square of RZ is to the rectangle under twice GB and ZQ, and, by alternating, the square of RX is to the square of RZ, as the rectangle under twice DA and XP is to the rectangle under twice GB and ZQ (16. 5 *Eu.*), or, in a ratio, compounded of the ratios of XP to ZQ and twice DA to twice GB (23. 6 *Eu.*) or, twice DA being to twice GB, as DA to GB (15. 5 *Eu.*), in a ratio compounded of the ratios of XP to ZQ and of DA to GB.

PROP. XIX. THEOR.

If, in a diameter of a parabola, any point be taken; the rectangle under the segments of the diameter, between its vertex and that point, and its vertex and the directrix, is to the difference of the squares, of the distances of the assumed point, from the focus, and from the directrix, as the square of the segment of the diameter, between its vertex and the directrix, is to the square of a right line, joining the focus, to the point, in which the diameter meets the directrix.

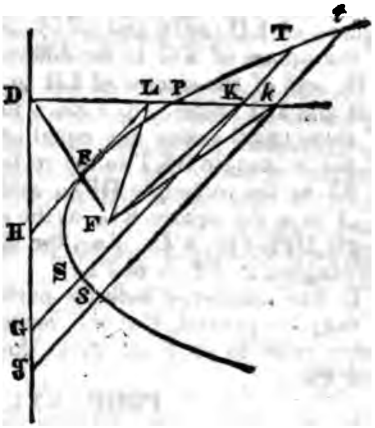


In a diameter DPK of a parabola, take any point whatever K; let P and D be the points in which the diameter meets the parabola and the directrix DG, and F the focus, and let DF, PF and KF be joined. The rectangle DPK is to the difference of the squares of KF and KD, as the square of PD is to the square of FD.

From K draw KM parallel to FP, meeting DF, produced, if necessary, in M. Because of the equiangular triangles DKM and DPF, KM is to KD, as PF to PD (4. 6 *Eu.*), or in a ratio of equality (Def. 8. 1 *Sup.*): from the centre K, at the distance KD, let a circle be described, which, because of the equality of KD and KM, passes through M; and, because of the parallels PF and KM, the rectangle DPK is to the rectangle DFM, or, which is equal (*Cor.* 3. 36. 3 *Eu.*), the difference of the squares of KF and KD, as the square of PD is to the square of FD (2 and 20. 6 and *Cor.* 3. 22. 5 *Eu*).

PROP. XX. THEOR.

If two right lines, parallel to each other, both cut in two points, or one of them touch, and the other so cut, a parabola, and meet a diameter; the rectangles under the segments of the secants, or square of the segment of the tangent, between the diameter, and the point or points, wherein they meet the parabola, are to each other, as the segments of the diameter, between its vertex and the parallels.



Let SKT and skt be two secants, or SKT a secant, and RL a tangent to a parabola, parallel to each other; the secants cutting the parabola in S and T, s and t, and meeting a diameter DK in K and k, and the directrix DG in G and g; and the tangent touching the parabola in R, and meeting the diameter in L, and the directrix in H. The rectangles SKT and skt, or the rectangle SKT and square or RL, are to each other, as the segments KP, kp and LP.

Join DF, KF, kF and LF; and first, let the parallel right lines which meet the diameter, be a secant, as ST, and a tangent, as RL; and since the determining ratio in the parabola is the ratio of equality [*Schol.* 6. 1 *Sup.*], the right lines KD, kD and LD have to the distances of the points K, k and L from the directrix, the determining ratio; whence the rectangle SKT is to the difference of the squares of KF and KD, as the square of KG is to the difference of the squares of KG and KD (13 1 *Sup.*); and the difference of the squares of KF and KD is to the rectangle DPK, as the square of DF is to the square of DP (19. 1 *Sup. and Theor.* 3. 15. 5 *Eu.*), therefore, by compounding these ratios, the rectangle SKT is to the rectangle DPK in a ratio compounded of the ratios of the rectangle SKT to the difference of the squares of KF and KD, and of the same difference to the rectangle DPK (*Def.* 13. 5 *Eu.*), or, which has been just shewn to be equal, of the ratios of the square of KG to the difference of the squares of KG and KD, and of the square of DF to the square of DP.

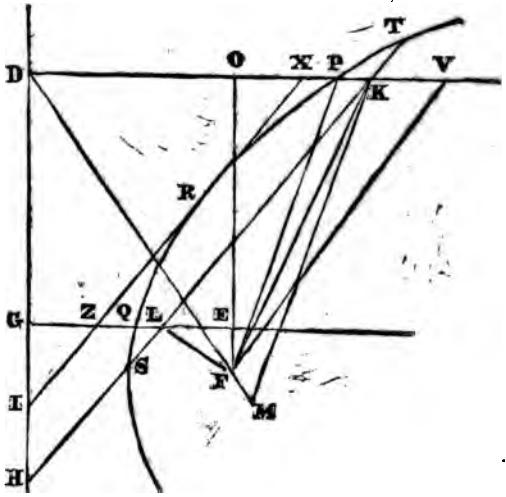
In like manner it may be proved, that the square of RL is to the rectangle DPL, in a ratio, compounded of the ratios of the square of LH to the difference of the squares of LH and LD, and of the square of DF to the square of DP.

But in these two compound ratios, that of the square of DF to the square of DP is common to both, and because of the equiangular triangles KGD and LHD, the ratios of KG to KD and of LH to LD are equal (4. 6 *Eu.*), and therefore those, of the square of KG to the square of KD, and of the square of LH to the square of LD (20. 6 and *cor.* 3. 22. 5 *Eu.*), and therefore those, of the square of KG to the difference of the squares of KG and KD, and of the square of LH to the difference of the squares of LH and LD (*Schol.* 18. 5 *Eu.*), therefore the ratios compounded of these equal ratios are equal (22. 5 *Eu.*), namely, the ratios of the rectangle SKT to the rectangle DPK, and of the square of RL to the rectangle DPL, and by alternating, the rectangle SKT is to the square RL, as the rectangle DPK is to the rectangle DPL (16. 5 *Eu.*), or, the side DP being common to both rectangles, as PK is to PL.

In like manner, if both the parallels be secants, as ST and st, it may be proved, that the rectangles SKT and skt have the same ratio to each other, as the segments of the diameter PK and Pk.

PROP. XXI. THEOR.

If a right line, touching a parabola, or cutting it in two points, meet two diameters; the squares of the segments of the tangent, or rectangles under the segments of the secant, between the diameters, and the point or points, wherein the tangent or secant meets the parabola, are to each other, as the segments of the diameters, between their vertices, and the points, in which they meet the tangent or secant.



Let DK and GL be two diameters of a parabola, meeting the section in P and Q, and the directrix in D and G, and let a tangent XZ, touching the parabola in R, meet these diameters in X and Z, and the directrix DG in I, or a secant ST, meet the same diameters in K and L, the section in S and T, and the directrix in H; the squares of RX and

RZ are to each other, as the segments of XP and ZQ; and the rectangles SKT, and SLT are to each other, as the segments PK and QL.

And first, the rectangle SKT and SLT are to each other, as the segments PK and QL; let F be the focus, and join FD, FP, FK and FL, let fall the perpendicular FO on DK, meeting GL in E, and on DK take OV equal to OD and join FV, through K, draw KM parallel to FP, meeting DF produced in M.

The right line FV is equal to FD (4. 1 *Eu.*), and PF to PD (Def. 8. 1 *Sup.*), therefore the triangles DFV and DPF are isosceles, and, having the angle PDF at the base of each common, are equiangular; therefore DV is to DF, as DF is to DP (4. 6 *Eu.*); or, which is equal, because of the parallels PF and KM (2. 6 *Eu.*), as FM is to PK; therefore the rectangle under DV, or twice DO and PK, is equal to the rectangle DFM (16. 6 *Eu.*): but, because of PF equal to PD, and parallel to KM, the right lines KD and KM are equal, and a circle described from the centre K, at the distance KD, would pass through M, and so the rectangle DFM is equal to the difference of the squares of KD and KF (*Cor.* 3. 36. 3 *Eu.*); therefore the rectangle under twice DO and PK is equal to the difference of the squares of KD and KF.

In like manner it may be proved, that the rectangle under twice GE or twice DO and QL, is equal to the difference of the squares of LG and LF.

But the rectangle SKT is to the difference of the squares of KD and KF, as the square of KH is to the difference of the squares of KH and KD (13, 1 *Sup.*), and the rectangle SLT is to the difference of the squares of LH and LF, as the square of LH is to the difference of the squares of LH and LG (by the same); and, because of the equiangular triangles KHD and LHG, the ratios of KH to KD and of LH to LG are equal (4. 6 *Eu.*), and therefore the ratios of the square of KH to the square of KD and of the square of LH to the square of LG (20. 6 *and Cor.* 3. 22. 5 *Eu.*), and therefore the ratios of the square of KH to the difference of the squares of KH and KD and of the square of LH to the difference of the squares of LH and LG (*Schol.* 18, 5 *Eu.*); therefore the ratios equal to them of the rectangle SKT to the difference of the squares of KD and KF, and of the rectangle SLT to the difference of the squares of LG and LF are equal; but the difference of the squares of KD and KF is above proved to be equal to the rectangle under twice DO and PK, and the difference of the squares of LG and LF, to the rectangle lelograms OC and CP, the right lines OK and KP are equal to LC and CH (34. 1 *Eu.*), therefore OK and KP are equal, and so OP is bisected in K.

Scholium. A perpendicular, let fall from any point of a conick section, on an axis, which is not the second axis of a hyperbola, meets the axis within the section, for it is parallel to a tangent, drawn through the nearer vertex of the axis (*Cor.* 1. 11. 1 *Sup. and* 28. 1 *Eu*), which tangent falling wholly without the section (*Def.* 10. 1 *Sup.*), if the perpendicular did not meet the axis within the section, it would meet the tangent, contrary to the definition of parallel right lines.

And this perpendicular is an ordinate to the axis (*Cor.* 2. 11. 1 *and Def.* 12. 1 *Sup.*), and if it be produced beyond the axis, so that the part produced may be equal to the ordinate, its other extreme is in the section, for otherwise, a right line ordinately applied to the axis, and terminated by the section, would not be bisected by the axis, contrary to this proposition.

Cor. A tangent to a conick section, which is perpendicular to an axis, which is not the second axis of a hyperbola, touches the section in a vertex of that axis; for a perpendicular to the axis, drawn from any other point of the section, meets the axis within the section, by the prec. schol. and would not therefore be a tangent (*Def.* 10. 1 *Sup*).

PROP. XXIII. THEOR.

If from any point of a conick section, an ordinate be drawn to an axis; the square of an ordinate is, in the case of an ellipse, or principal axis of a hyperbola, to the rectangle under the abscissas, and, in the case of the second axis of a hyperbola, to the sum of the squares of the second semiaxis, and the segment thereof between the centre and ordinate, as the square of the semiaxis to which the ordinate is parallel, is to the square of the other; and, in the case of a parabola, the square of the ordinate is equal to the rectangle, under the abscissa, and the principal parameter.

Part 1. When the figure is an ellipse, (see fig. 1 preceding prop.), PH being an ordinate to the axis AB, and PK to the axis MN; these ordinates are perpendicular to their respective axes (*Cor.* 3. 11. 1 *Sup.*), and it is manifest from the 1st, 14th and 22. 1 Sup. that the square of PH is to the rectangle AHB, as the square of CM is to the square of CB, and the square of PK to the rectangle NKM, as the square of CB to the square of CM.

Part 2. When the ordinate, as PH, (*see fig.* 2. *preceding prop.*), meets the principal axis AB of a hyperbola. The square of PH is to the rectangle AHB, as the square of CM is to the square of CB.

From the focus F, draw FD at right angles to AH, meeting the hyperbola in D, draw DG at right angles to the adjacent directrix TS, which let AB meet in Z.

FD is to DG or FZ, as FB is to BZ (6. 1 *Sup.*), and by alternating, FD is to FB, as FZ is to BZ (16. 5 *Eu.*); but since CF is to CB, as FB is to BZ (*Schol.* 6. 1 *Sup.*), by compounding, CF and CB together, or AF is to CB, as FB and BZ together, or FZ is to BZ (18. 5 *Eu.*); but it is above shewn, that FD is to FB, as FZ to BZ; therefore FD is to FB, as AF is to CB (11. 5 *Eu.*), and therefore the rectangle under FD and CB is equal to the rectangle AFB (16. 6 *Eu.*), or, which is equal (2. 1 *Sup.*), to the square of CM; therefore CM is a mean proportional between CB and FD, and so the square of FD is to the square of CM, or (2. 1 *Sup.*), the rectangle AFB, as the square of CM is to the square of CB (20. 6 *and Cor.* 3. 22. 5 *Eu.*), but the square of PH is to the rectangle AHB, as the square of FD is to the rectangle AFB (14 and 22. 1 *Sup.*), therefore the ratios of the square of PH to the rectangle AHB, and of the square of CM to the square of CB, being each equal to that of the square of FD to the rectangle AFB, are equal to each other (11. 5 *Eu*).

Part 3. When the ordinate, as PK, meets the second axis MN of a hyperbola. The square of PK, is to the sum of the squares of CM and CK, as the square of CB is to the square of CM.

For, (*by the prec. part and inverting*), the rectangle AHB is to the square of PH or CK, as the square of CB is to the square of CM; therefore the rectangle AHB with the square of CB, or, which is equal (6. 2 *Eu.*), the square of CH or PK, is to the square of CM and CK together, as the square of CB is to the square of CM (12. 5 *Eu*).

Part 4. When the figure is a parabola, see fig. 3 of prec. prop. The square of PH, is equal to the rectangle under BH, and the principal parameter.

From the focus F draw FD at right angles to BH, meeting the parabola in D, let fall the perpendicular DG on the directrix ST, which let the axis BH meet in Z.

The right line DF is equal to DG (*Def.* 8. 1 *Sup.*), therefore the square of DF is equal to the square of GD or of ZF, or, ZB and BF being equal (*Def.* 8. 1 *Sup.*), to four times the square of

BF (*Cor.* 4. 2 *Eu.*), or to the rectangle under BF and four times BF, or, the principal parameter being equal to four times BF (*Def.* 16. *and* 17. 1 *Sup.*), to the rectangle under BF and the principal parameter; but the square of DF is to the square of PH, as BF is to BH (22 *and* 20. 1 *Sup.*), or, as the rectangle under BF and the principal parameter, is to the rectangle under BH and the same parameter (1. 6 *and* 11. 5 *Eu.*); whence, the square of DF having been just proved equal to the rectangle under BF and the principal parameter, the square of PH is equal to the rectangle under BH and the same parameter (14. 5 *Eu*).

Cor. 1. Hence, in ellipses, the rectangle under the abscissas of the greater axis is greater, of the second axis, less, than the square of the ordinate.

Cor 2. And, in ellipses and hyperbolas, a right line, drawn from the centre, at right angles to the principal axis, whose square, is to the square of the principal semiaxis, as the square of an ordinate to the principal axis, is to the rectangle under the abscissas, is the second semiaxis.

PROP. XXIV. THEOR.

A right line, terminated by a conick section, passing through a focus, and ordinately applied to the principal axis, is equal to the principal parameter.

Fig. 1.

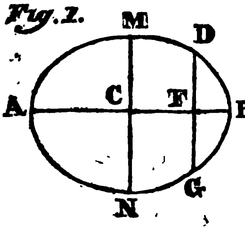


Fig. 2.

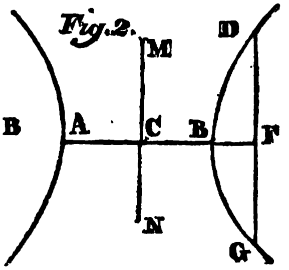
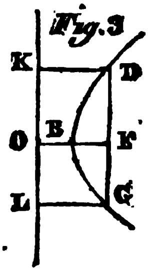


Fig. 3.



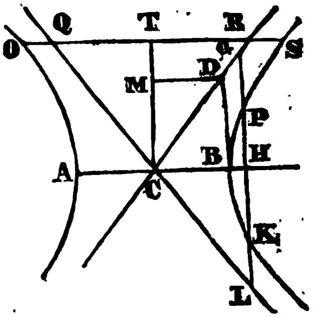
Let BD be a conick section, see fig. 1, 2 and 3, whose principal axis is BF, and DG a right line passing through a focus F, ordinately applied to the same axis, and terminated both ways by the section; DG is equal to the principal parameter of the section.

First, let the section be an ellipse or hyperbola, see fig. 1 and 2, let A and B be the principal vertices, C the centre, and MN the second axis. The square of CB is to the square of CM, as the rectangle AFB, or, which is equal (2. 1 *Sup.*), the square of CM, is to the square of FD (23. 1 *Sup.*); therefore the right lines CB, CM and FD are continually proportional (22. 6 *Eu.*), and therefore also their doubles (1 *&* 22. 1 *Sup.*), AB, MN and DG (15. 5 *Eu.*); whence, AB and MN being conjugate diameters (*Def.* 14. 3 *and* 5, *and Cor.* 1. 11. 1 *Sup.*), DG is equal to the parameter of the principal axis AB, or the principal parameter of the section (*Def.* 15. *and* 17. 1 *Sup*).

Let now DG, see fig. 3, be a right line, terminated by a parabola, passing through its focus F, and ordinately applied to its axis BF; let KL be the directrix, which let FB produced, meet in O, and draw DK and GL at right angles to KL; BF is equal to OB (*Def.* 8. 1. *Sup.*), and therefore OF, or either of its equals (34. 1 *Eu.*), DK or GL, double to OB; whence, FD being equal to DK, and FG to GL (*Def.* 8. 1 *Sup*), DG is fourfold of OB or BF, and therefore equal to the principal parameter (*Def.* 16. *and* 17. 1 *Sup*).

PROP. XXV. THEOR.

If a right line, touching a hyperbola, or cutting a hyperbola or opposite hyperbolas, and parallel to either axis, meet both asymptotes; the square of the segment of the tangent, between the hyperbola, and either asymptote, or rectangle under the segments of the secant, between an intersection with the hyperbola, or either of the opposite hyperbolas, and the asymptotes, or between either asymptote, and the hyperbola or opposite hyperbolas, is equal to the square of the semiaxis, which is parallel to the tangent or secant.



Part. 1. Let a tangent BD, to a hyperbola BP, touching the section in B, parallel to the second axis CM, meet the asymptote CR in D; the square of BD is equal to the square of the semiaxis CM.

Because the tangent BD is parallel to CM (*Hyp.*), it is perpendicular to the transverse axis AB (*Def.* 5. 1 *Sup.* and 29. 1 *Eu.*), and therefore its contact is in its vertex B (*Cor.* 22. 1 *Sup.*), and BD is equal to CM (*Def.* 19. 1 *Sup.*); therefore the square of BD is equal to the square of CM.

Part 2. Let GL parallel to the second axis CM, meet the transverse axis in H, the hyperbola in P and K, and the asymptotes in G and L; the rectangle GPL or PGK, is equal to the square of CM.

For the triangles GCH and LCH, having the angles at C equal (*Cor. Def.* 19. 1 *Sup.*), the angles at H equal, being right, and CH common, GH is equal to HL (26. 1 *Eu.*); and PK is bisected in H (*Cor.* 2. 11. 1 and 22. 1 *Sup.*), and therefore, taking equals from equals, GP is equal to KL; and, because of the equiangular triangles CBD and CHG, the square of CH is to the square of HG, as the square of CB to the square BD (4 and 22. 6 *Eu.*), or (*by part* 1 *of this prop.*), of CM, and therefore (23. 1 *Sup.*), as the rectangle AHB is to the square of HP; therefore, the excess of the square of CH above the rectangle AHB, or, which is equal (6. 2 *Eu.*), the square of CB, is to the excess of the square of HG above the square of HP, or, which is equal (5. *and* 6. 2 *Eu.*), the rectangle GPL or PGK, as the square of CH to the square of HG (19. 5 *Eu.*), or, which is equal (4 and 22. 6 *Eu.*), as the square of CB to that of BD or CM; since then the square of CB has the same ratio to the rectangle GPL or PGK and the square of CM, the rectangle GPL or PGK is equal to the square of CM (95. *Eu*).

Part 3. Let the secant OS, parallel to the transverse axis AB, meet the opposite hyperbolas in O and S, and the asymptotes in Q and R, the rectangle RSQ or ORS is equal to the square of CB.

Let the second semiaxis CM meet OS in T, draw MD from M at right angles to CM, meeting CG in D; MD is equal to

CB (*Def.* 19. 1 *Sup.*), and since, because of the equiangular triangles CTR and CMD, the square of CT is to the square of TR, as the square of CM to the square of MD (4. *and* 22. 6 *Eu.*), or of CB, or, which is equal (23. 1 *Sup.*), as the squares of CM and CT together to the square of TS ; therefore the excess of the sum of the squares of CM and CT above the square of CT, or, the square of CM, is to the excess of the square of TS above that of TR, or, to that which is equal by 5 and 6. 2 Eu. both QR and OS being bisected in T (26. 1 *Eu. and* 22. 1 *Sup.*), the rectangle RSQ or ORS, as the square of CT is to the square or of TR (19. 5 *Eu.*), or, which is equal (4 *and* 22. 6 *Eu.*), as the square of CM is to the square of MD or CB ; whence, the rectangle RSQ or ORS and the square of CB, to each of which the square of CM has the same ratio, are equal (9. 5 *Eu*).

PROP. XXVI. THEOR.

A hyperbola (BP, see preceding fig.), and its asymptote (CG), may be so produced towards the part (GP) remote from the centre, that the hyperbola would approach nearer to the asymtote, than by any given distance, but cannot, at any finite distance from the centre, coincide with, or meet it.

Through any point P of the hyperbola, let GL be drawn parallel to the second semiaxis CM, meeting the asymptotes in G and L, and the transverse axis AB in H ; and since GPL is always of the same magnitude, wherever in the hyperbola the point P be taken, being always equal to the squares of CM (25. 1 *Sup.*), and since the square of PH has a constant ratio to the rectangle BHA (23. 1 *Sup.*), and the point H may be so taken, that the rectangle BHA may be greater than any given square, by taking BH greater than the side of that square ; it follows, that the point H may be so taken, that the square of PH may be greater than any given square, and PH, and of course PL greater than any given right line, and therefore GP, and of course a perpendicular let fall from P on CG, being the distance of P from CG, less than any given distance, or given finite right line.

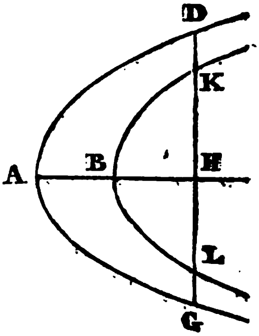
And because of the given magnitude of the rectangle GPL, the point P cannot, at any finite distance from the centre C coincide with the asymptote CG.

Scholium. Hence the language of Mathematicians, when they speak of asymptotes of a hyperbola, as tangents tending to a point infinitely distant.

Cor. Hence any right line drawn from the centre of a hyperbola within two asymptotes CG, CL of a hyperbola, meets the hyperbola SBK which is within the asymptotes.

PROP. XXVII. THEOR.

Two parabolas (DAG and KBL), having a common axis (AH), and equal principal parameters, and being towards the same part (DG) with respect to their principal vertices (A and B), may be so produced, as to approach nearer than by any given distance, but cannot, at any finite distance from their principal vertices, coincide or meet.

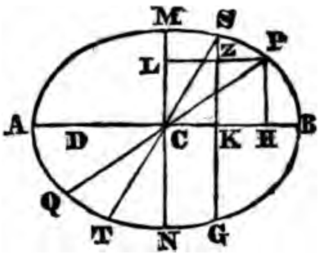


From any point K, of the interior parabola BK, let DG be drawn perpendicular to the axis AH, and of course ordinately applied to it (*Cor.* 2. 11. 1 *Sup.*), meeting the axis in H, the exterior parabola in D and G, and the interior again in L.

Because the square of DH is equal to the rectangle under AH and the common principal parameter (23. 1 *Sup.*), and the square of KH equal to the rectangle under BH and the same parameter (*by the same*), the difference of these squares of DH and KH, or, which is equal (22. 1. *Sup. and* 5. 2 *Eu.*), the rectangle DKG, is equal to the rectangle under AB and the same parameter (1, 2 *Eu.*), and therefore of the same magnitude, through whatever point of the interior parabola, the right line DG be drawn; since then the point K may be so taken, that BH, and therefore KH whose square increases in the same ratio as the right line BH does 23. 1 *Sup. and* 1, 6 *Eu.*), and therefore KG, would be greater than any given right line, it may also be so taken, that DK would be less than any given right line. But the points D and K cannot, at any given finite distance, coincide, because of the given magnitude of the rectangle DKG.

PROP. XXVIII. THEOR.

The transverse axis of an ellipse is the greatest, and the second axis the least, of all its diameters; and, of others, the nearer to the transverse axis, is greater than the more remote.



Let AB be the transverse, and MN the second axis of an ellipse, and QP and TS other diameters, of which QP is the nearer to AB.—The transverse axis AB is the greatest, and the second axis MN the least, of all the diameters, and QP, the nearer to AB, is greater than the more remote TS.

Draw PH and SK at right angles to AB, and let SK produced, meet the ellipse again in G, draw PL at right angles to MN, meeting SG in Z, and take AD equal to HB; and since the rectangle AHB is greater than the square of PH (*Cor.* 1. 23. 1 *Sup.*), adding to each the square of CH, the rectangle AHB with the square of CH, or, which is equal (5. 2 *Eu.*), the square of CB, is greater than the squares of CH and HP together, or, which is equal (47. 1 *Eu.*), the square of CP, and so CB is greater than CP, and AB and QP being their doubles (1 *and* 5. 1 *Sup.*), AB is greater than QP. Therefore AB is the greatest of all the diameters.

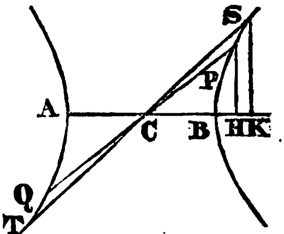
And since the rectangle MLN is less than the square of LP (*Cor.* 1. 23. 1 *Sup.*), adding to each the square of CL, the rectangle MLN with the square of CL, or, which is equal (5. 2 *Eu.*), the square of CM, is less than the square of CL and LP, or (47. 1 *Eu.*), the square of CP, and so CM is less than CP, and MN than QP. Therefore MN is the least of all the diameters.

And since the rectangle AKB, or the difference of the squares of CB and CK, is to the square of SK, as the rectangle AHB, or the difference of the squares of CB and CH, is to the square of PH (23. 1 *Sup. and* 11. 5 *Eu.*), the rectangle AKB is to the square of SK, as the difference of the rectangles AKB and AHB is to the difference of the squares of SK and PH (19. 5 *Eu.*); but the difference of the squares of AKB and AHB is equal to the rectangle DKH (*Cor.* 2. 5. 2 *Eu.*), and the difference of the squares of SK and PH or of SK and ZK to the rectangle SZG (5. 2 *Eu.*); therefore the rectangle AKB is to the square of SK, as the rectangle DKH is to the rectangle SZG;

but the rectangle AKB is greater than the square of SK (*Cor.* 1. 23. 1 *Sup.*), therefore the rectangle DKH is greater than the rectangle SZG (*Cor.* 13. 5 *Eu.*); adding to each the squares of CK and HP, the rectangle DKH with the squares of CK and HP, is greater than the rectangle SZG with the squares of CK and HP; but the rectangle DKH with the squares of CK and HP is equal to the squares of CH and HP (5. 2 *Eu.*), or, which is equal (47. 1 *Eu.*), the square of CP, and the rectangle SZG with the squares of CK and HP or of CK and KZ, is equal to the squares of CK and KS (5. 2 *Eu.*), or, which is equal (47. 1 *Eu.*), the square of CS; therefore the square of CP is greater than the square of CS, and therefore CP than CS, and QP than TS.

PROP. XXIX. THEOR.

Of all the diameters, terminated by opposite hyperbolas, the axis is the least; and, of others, the nearer to the axis is less than the more remote.



Let AB be the axis, and QP and TS other diameters, terminated by opposite hyperbolas AT and BS, the diameter QP being nearer to AB, than TS.—AB is the least of all the diameters so terminated, and QP is less than TS.

Let fall the perpendiculars PH and SK on AB produced; the square of CP is equal to the squares of CH and HP (47. 1 *Eu.*), and therefore greater than the square of CH, or its part CB, and so the right line CP is greater than CB, and AB and QP being double of CB and CP (1 *and* 5. 1 *Sup.*), QP greater than AB, and so AB is the least of all diameters terminated by the opposite hyperbolas AT and BS.

And since the rectangle AKB is to the square of SK, as the rectangle AHB is to the square of PH (23. 1 *Sup. and* 11. 5 *Eu.*), and the rectangle AKB is greater than the rectangle AHB, the square of SK is greater than the square of PH (14. 5 *Eu.*); whence, the square of CK being greater than the square of CH, the squares of CK and KS together, or, which is equal (47. 1 *Eu.*), the square of CS, is greater than the squares of CH and HP together, or (47. 1 *Eu.*), the square of CP; therefore CS is greater than CP, and, QP and TS being double to CP and CS (5. 1 *Sup.*), TS is greater than QP.

PROP. XXX. THEOR.

Tangents (PD and QG, see fig. 1 *and* 2*), at the vertices (P and Q), of any diameter (PQ), of an ellipse or hyperbola, are parallel to each other.*

Fig. 1.

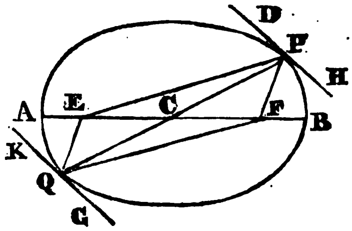
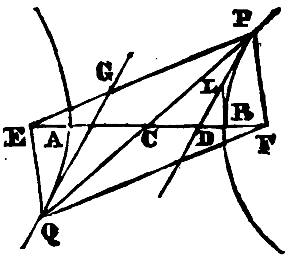


Fig. 2.



Let AB be the principal axis of the ellipse or hyperbola, C the centre, and E and F the focuses; join EP, PF, EQ and QF, and produce in fig. 1, DP and GQ, as to H and K.

Because, in the triangles ECP and FCQ, EC is equal to CF (*Def.* 3. *and* 5. 1 *Sup*), CP to CQ (5. 1 *Sup*.), and the angles ECP and FCQ also equal (15. 1 *Eu*.), the angles CPE and CQF are equal (4. 1 *Eu*.); in like manner, the angles CPF and CQE may be proved equal; therefore, the aggregates of these equal angles, namely, the angles EPF and EQF are equal (*Ax*. 2. 1 *Eu*).

Whence, because, in fig. 1 the angles EPD and FPH are equal (11. 1 *Sup*. *and* 15. 1 *Eu*.), the angle EPD is half the complement of the angle EPF to two right angles; in like manner, it may be proved, that FQG is half the complement of the angle EQF to two right angles; therefore the angles EPD and FQG are equal (*Ax*. 3. *and* 7. 1 *Eu*.); to which adding the equal angles EPC and FQC, the angles DPQ and PQG are equal (*Ax*. 2. 1 *Eu*.), and these, the right line PQ meeting the right lines DP and QG, are alternate angles, therefore DP and QG are parallel (27. 1 *Eu*).

In like manner, in fig. 2, the angles EPD and FQG, being halves of the equal angles EPF and EQF (11. 1 *Sup*.), are equal (*Ax*. 7. 1 *Eu*.); taking from them the equal angles EPC and and FQC, the angles DPQ and PQG are equal (*Ax*. 3. 1 *Eu*.), and therefore DP and QG are parallel (27. 1 *Eu*).

Cor. 1. A tangent, passing through the vertex of a diameter, which is not an axis, is not perpendicular to the diameter.

In fig. 1 above, if P be not the vertex of the second axis, the angles ECP and FCP are unequal [*Def.* 3. 1 *Sup.*], let ECP be the greater, and the triangles ECP and FCP, having the sides about these unequal angles equal, EP is greater than PF [24. 1 *Eu.*], and by a similar reasoning to that used in the case of the hyperbola in prop. 5. 1 *Sup.* the angle EPC may be proved to be less than FPC, adding to them the equal angles EPD and FPH, the angle CPD is less than CPH, and so the tangent DH is not perpendicular to the diameter QP.

In fig. 2, let the tangent PD meet the principal axis AB in D, and at the point B draw BL perpendicular to AB, and of course touching the section in B [*Schol.* 10. 1 *Sup.*], it must therefore meet the tangent PD between the points P and D, as in L, and the external angle CDP of the triangle DBL is greater than the internal remote angle DBL [16. 1 *Eu.*] and therefore obtuse, and so the angle CPD acute [*Cor.* 17. 1 *Eu.*], therefore PD is not perpendicular to the diameter QP.

In the case of a parabola, see fig. 3. 10. 1 Sup, the angle HPK, which the diameter HP makes with the tangent PK, is half of the angle HPF [11. 1 *Sup.*], whence, the angle HPF being less than two right angles, HPK is less than a right angle. Therefore in every case, a tangent, passing through the vertex of a diameter, which is not an axis, is not perpendicular to the diameter.

Cor. 2. A diameter of a conick section, which is perpendicular to a tangent passing through its vertex, is an axis.

For if it were not an axis, it would not be perpendicular to a tangent passing through its vertex, by the preceding corollory.

PROP. XXXI.

A diameter of a conick section bisects all right lines terminated by the section or opposite sections, and ordinately applied thereto.

Fig. 1.

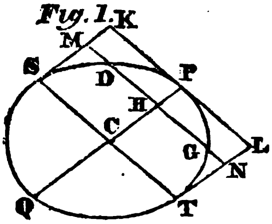


Fig. 2.

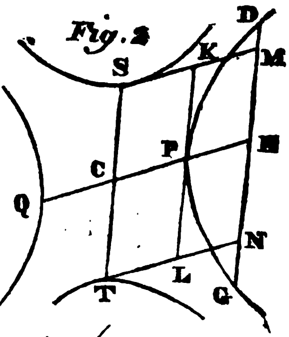


Fig. 3.

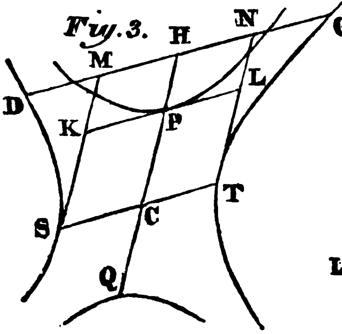
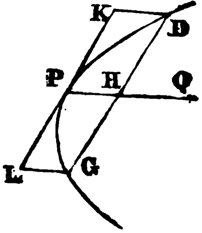


Fig. 4.



Let PQ, see fig. 1, 2 and 3, be a diameter of a conick section, DG a right line terminated by the section, and ordinately applied to the diameter PQ, and let PQ meet DG in the point H.—DG is bisected in H.

Let KL be a tangent to the section at the vertex P of the diameter PQ, which is parallel to DG [*Def.* 12. 1 *Sup*).

In fig. 1, 2 *and* 3, let ST be a diameter of the section parallel to DG, and S and T its vertices, and SK and TL tangents drawn through these vertices, in the case of fig. 1, to the section, of fig. 2, to the conjugate, and of fig. 3, to the opposite sections, meeting the tangent KL in K and L, and DG in M and N; the tangents SK and TL are parallel to each other [30. 1 *Sup.*); and in the parallelograms SN, SH and CN, the right line SM is equal to TN, MH to SC and HN to CT [34. 1 *Eu.*], whence ST being bisected in C [5. 1 *Sup.*], MN is bisected in H; but the square of SM is to the square of TN, as the rectangle DMG is to the rectangle DNG [*Cor.* 5. 14. 1 *Sup*]; whence, the squares of SM and TN being, because of the equality of the right lines themselves, equal, the rectangles DMG and DNG are equal [9. 5 *Eu.*], therefore MD and GN are equal [*Cor.* 1 *and* 2. 7. 2 *Eu.*); whence, MH and HN being equal, as has been just shewn, by taking from them in fig. 1, and adding to them in fig. 2 and 3, the equals DM, GN, the sums or differences, DH and HG are equal, and so DG is bisected in H.

In fig. 4, let KL be a tangent to a parabola, touching it in the vertex P of the diameter PQ ; KL is parallel to DG [*Def.* 12. 1 *Sup.*] ; through D and G draw DK and GL parallel to the diameter PQ, and of course themselves diameters [*Def.* 9. 1 *Sup.*], meeting KL in K and L ; and, because of the parallelogram KG, KD is equal to LG [34. 1 *Eu.*] ; but the square of PK is to the square of PL, as KD is to LG, [21. 1 *Sup.*], or in a ratio of equality, therefore the right lines LP and PK themselves are equal, and therefore also the right GH and HD, which are, because of the parallelograms LH and PD, equal to LP and PK.

PROP. XXXII. THEOR.

A right line (DG, see fig. 1, 2, 3 and 4 of the prec. prop.), not being a diameter, which is terminated by a conick section, or opposite sections, and bisected by a diameter (QP, as in H), is ordinately applied to the diameter.

Let KL be a tangent, drawn through a vertex P of the diameter QP, and, in fig. 1, 2 and 3, let ST be a diameter parallel to KL, let SK and TL be tangents to the section or opposite or conjugate sections, as the case may be, drawn through the vertices S and T, meeting the tangent KL in K and L, and DG, produced if necessary, in M and N.

Because ST is parallel to KL [*constr.*], and SK to TL [30. 1 *Sup.*], STLK is a parallelogram, and so SK is equal to TL, and KL to ST [34. 1 *Eu.*] ; but the square of SK is to the square of TL, as the square of KP to the square of PL [*Cor.* 2. 14. 1 *Sup.*], therefore SK is to TL, as KP to PL [22. 6 *Eu.*], whence, SK being equal to TL, KP is equal to PL [*Cor.* 13. 5 *Eu.*], and therefore ST, which is bisected in C [5. 1 *Sup.*], being equal to KL, the right lines KP and PL are each of them equal to SC or CT, and being parallel to them, CP is parallel to SK or TL ; whence, because of the equals SC and CT, and the parallels SM CH and TN, the right line MH is equal to HM [*Cor.* 1. 9. 1 *Sup.*], but DH is equal to HG [*Hyp.*], therefore MD is equal to GN and MG to DN (*Ax.* 2 *and* 3. 1 *Eu.*), therefore the rectangle DMG is equal to the rectangle GND (*Cor.* 3. 34. 1 *Eu.*), but the square of SM is to the square of TN, as the rectangle DMG is to the rectangle GND (*Cor.* 5. 14. 1 *Sup.*); therefore the square of SM is equal to the square of TN, and so SM is equal to TN, which right lines being parallel, MN is parallel to ST (33. 1 *Eu.*), and therefore to KL (*Hyp. and* 30. 1 *Eu.*), whence DG, being parallel to the tangent KL drawn through the vertex P of the diameter QP, is ordinately applied to that diameter (*Def.* 12. 1 *Sup*).

And, in the case of a parabola, see fig. 4, having drawn DK and GL parallel to the diameter PQ, meeting the tangent KL in K and L; because KD, PH and LG are parallel (*Constr.*), and the right line GH equal to HD (*Hyp.*), LP is equal to PK (*Cor.* 1. 9. 1 *Sup.*), and therefore the square of LP to the square of PK; but the square of LP is to the square of PK, as LG is to KD (21. 1 *Sup.*); whence, the squares of LP and PK being equal, the right lines LG and KD are equal, and being parallel, DG is parallel to KL (33. 1 *Eu.*), and of course ordinately applied to the diameter PQ (*Def.* 12. 1 *Sup*).

Cor. 1. Hence, if two or more parallel right lines be terminated by a conick section or opposite sections, the diameter which bisects one of them, not being a diameter, bisects them all.

For that which is supposed to be bisected, is ordinately applied to the diameter bisecting it (*by this prop.*), therefore the others are ordinately applied to the same diameter (*Hyp.* 30. 1 *Eu. and Def.* 12. 1 *Sup.*), and of course bisected by that diameter (31. 1 *Sup*).

Cor. 2. A right line, which bisects two parallel right lines terminated by a conick section or opposite sections, is a diameter.

For, if any other right line bisecting one of them were a diameter, it would bisect the other also (*by the preceding cor.*), which would therefore be bisected in two points, which is absurd.

Cor. 3. Two right lines terminated by a conick section or opposite sections, except right lines, which, in the case of an ellipse or hyperbola, are diameters, do not mutually bisect each other.

If, in the case of an ellipse or hyperbola, one of the right lines so terminated be a diameter, it is manifestly not bisected by the other, not being a diameter; if neither be a diameter, they do not mutually bisect each other, for if they did, they would both be ordinately applied to the diameter drawn through their concourse (*by this prop.*), and therefore parallel to each other (*Def.* 12. 1 *Sup. and* 30. 1 *Eu.*), which is absurd.

Cor. 4. A diameter (AB or MN, see fig. 1 and 2. 22. 1 Sup.), of an ellipse or hyperbola, which bisects a right line (PQ or OP) joining the vertices of two equal diameters, (OQ and RP) is an axis.

For the diameters OQ and RP being equal (*Hyp.*), the semidiameters CP and CQ, which are half the diameters (5. 1 *Sup.*), are equal; whence the triangles CHP and CHQ, having also PH equal to HQ (*Hyp.*), and CH common, have the angles at H equal (8. 1 *Eu.*), and therefore right; and PQ, being bisected in H (*Hyp.*), is ordinately applied to the diameter AB (32. 1 *Sup.*); whence the diameter AB, being perpendicular to its ordinate PH, is perpendicular to a tangent drawn through its vertex (*Def.* 12. 1 *Sup. and* 29. 1 *Eu.*), and is therefore an axis (*Cor.* 2. 30. 1 *Sup.*). In like manner MN may be proved to be an axis.

PROP. XXXIII. PROB.

To draw the diameter, to which a right line, terminated by a given conick section or opposite sections, is ordinately applied; and to find the centre of a given ellipse or opposite hyperbolas.

Fig. 1.

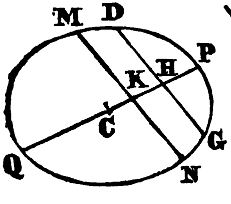


Fig. 2.

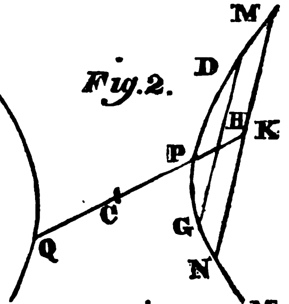


Fig. 3.

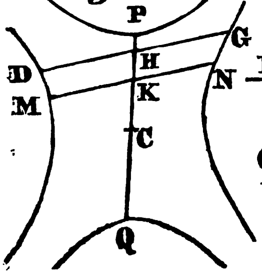
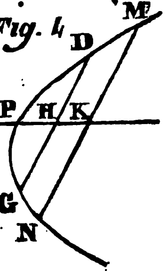


Fig. 4.



Part 1. Let DG, see fig. 1, 2, 3 and 4, be a right line, terminated by a given conick section or opposite sections. It is required to draw the diameter, to which DG is ordinately applied.

Draw a right line MN, parallel to DG, and terminated by the section or opposite sections, as the case may be, bisect DG and MN in H and K, join HK, which is a diameter of the section (*Cor.* 2. 32. 1 *Sup.*), and, since it bisects DG, is the diameter to which DG is ordinately applied (32. 1 *Sup.*), as was first required to be found.

Part 2. An ellipse or opposite hyperbolas being given, see fig. 1 and 2; it is required to find the centre.

Draw any two right lines DG and MN, terminated by the ellipse or one of the opposite hyperbolas, bisect these right lines in H and K, join HK, and produce it to meet the section or opposite sections in P and Q, bisect PQ in C; the point C is the centre of the section.

For because QPK bisects the right lines DG and MN, terminated by the section, it is a diameter (*Cor.* 2. 32. 1), and since every diameter of an ellipse or hyperbola is bisected in the centre (5. 1 *Sup.*), C is the centre of the section; for otherwise QP would be bisected in two points, which is absurd.

PROP. XXXIV. PROB.

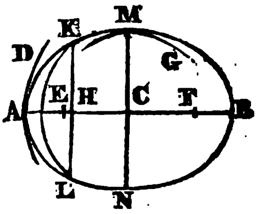
To draw a right line, touching a given conick section, parallel to a given right line terminated by the section; and a right line terminated by opposite hyperbolas being given, to draw a tangent to a given conjugate hyperbola, parallel thereto.

Draw the diameter to which the given terminated right line is ordinately applied (33. 1 *Sup.*), which, if the figure be a parabola, meets the section in its vertex, if not, let it meet the ellipse or given hyperbola in one of its vertices, a right line drawn through the vertex of the diameter, parallel to the given terminated right line, is a tangent to the section (*Def.* 12. 1 *Sup*).

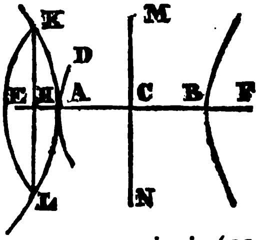
In the case of the given right line being terminated by opposite hyperbolas, the diameter found, meets the given conjugate hyperbola in one of its vertices, a right line drawn through which, parallel to the given right line, is a tangent to the section (*Def.* 12. 1 *Sup*).

PROP. XXXV. PROB.

To find the axes and focuses, or, if there be but one of each, that one, of a given conick section or opposite sections.

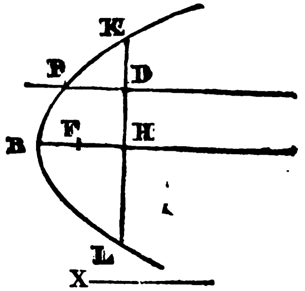


First, let the given figure be an ellipse, of which find the centre C (33. 1 *Sup.*), from which as a centre, through some point of the ellipse, describe a circle; if this circle fall entirely without the ellipse, as AD, AC is the greatest of the semidiameters, and therefore the greater semiaxis (28. 1 *Sup.*); if it fall entirely within the ellipse, as MG, CM is the least of the semidiameters, and therefore the second semiaxis (28. 1 *Sup.*); but if it be described through some point which is not a vertex of an axis, as K between A and M, it will necessarily meet again the ellipse, as in L, join KL, which bisect in H, and join CH, which is at right angles to KL (3. 3 *Eu.*); whence KL being ordinately applied to the diameter CH (32. 1 *Sup.*), and at right angles to it, the diameter CH is an axis (*Def.* 12. 1 *and Cor.* 2. 30. 1 *Sup.*), and being produced both ways meets the ellipse in its vertices A and B; through C, draw MN at right angles to AB, meeting the ellipse in M and N; MN is the other axis: divide the greater axis AB in E and F, so that the rectangle AEB or AFB may be equal to the square of CM or CN (*Cor.* 2. 6. 2 *Eu.*): E and F are the focuses of the ellipse (*Cor.* 1. 2. 1 *Sup*).



In like manner, in the case of a hyperbola, find the centre C (33. 1 *Sup.*), from which as a centre, through some point of one of the opposite hyperbolas, describe a circle; if this circle fall entirely without the hyperbola, as the circle AD does without the hyperbola AK, meeting it only in the point A, CA is the least of all the transverse semidiameters, and is therefore the transverse semiaxis (29. 1 *Sup.*); but if it be described through any other point of the hyperbola, as K, it meets the hyperbola again, as in L; join KL, which bisect in H, and join CH, which is at right angles to KL (3. 3. *Eu.*); whence, KL being ordinately applied to the diameter CH (32. 1 *Sup.*), and at right angles to it, the diameter CH is the transverse axis (*Def.* 12. 1. *and Cor.* 2. 30. 1 *Sup.*); let it meet the opposite hyperbolas in A and B, which are its vertices ; draw CM at right angles to AB, whose square may be to the square of CA, as the square of KH is to the rectangle AHB (*Cor.* 3. 23. 6 *Eu.*), CM is the second semiaxis (*Cor.* 2. 23. 1 *Sup.*) ; in AB produced both ways take the points E and F, so that the rectangles AEB and AFB may be each of them equal to the square of CM (*Cor.* 3. 6. 2 *Eu.*), the points E and F are the focuses of the opposite hyperbolas (*Cor.* 2. 2. 1. *Sup*).

In the case of a parabola, having found a diameter PD (33. 1 *Sup.*), through any point therein D, draw KL, at right angles to PD, meeting the parabola in K and L; bisect KL in H, and through H draw HB at right angles to KL, meeting the parabola in B ; BH being parallel to the diameter PD, is itself a diameter (*Def.* 9. 1 *Sup.* and 29. 1 *Eu.*), and KL an ordinate to it (32. 1 *Sup.*) ; whence the diameter BH, being perpendicular to its ordinate, is the axis (*Def.* 12. 1 *and Cor.* 2. 30. 1 *Sup.*) ; and since the square of KH is equal to the rectangle under BH and the principal parameter (23. 1 *Sup.*), a right line X, being taken a third proportional to BH and HK (11. 6 *Eu.*), is the principal parameter (23. 1 *Sup. and* 17. 6 *Eu.*) ; on BH take BF equal to a fourth part of X, F is the focus (*Def.* 16 *and* 17. 1 *Sup*).



Cor. Hence it appears, how a diameter of a given conick section may be drawn through a given point, which is not in an asymptote of a hyperbola ; namely, in the case of an ellipse or hyperbola, by finding the centre (*by this prop.*), and drawing a right line through that and the given point, which is a diameter (*Def.* 3 *and* 5. 1 *Sup.*) ; and, in the case of a parabola, by finding the axis (35. 1 *Sup.*), and drawing through the given point a right line parallel thereto, which is a diameter (*Def.* 9. 1 *Sup. and* 29. 1 *Eu*).

PROP. XXXVI. PROB.

To draw a right line, ordinately applied to a given diameter of a conick section or opposite sections, from a given point in the section or sections, not being a vertex of the given diameter.

Fig. 1.

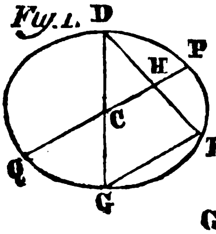


Fig. 2.

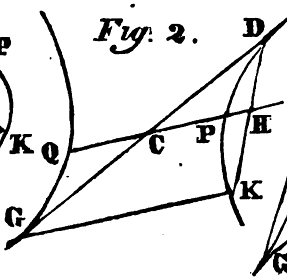
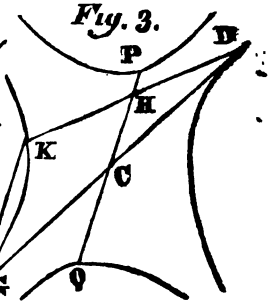


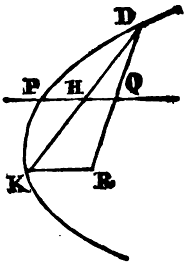
Fig. 3.



First, let the conick section be an ellipse or opposite hyperbolas; QP, see fig. 1, 2 and 3, being the given diameter, and D the point, from which the ordinate is required to be drawn.

Find the centre C [33. 1 *Sup.*], join DC, on which produced take CG equal to CD; G is the other vertex of the diameter DG [5. 1 *Sup.*]; through G, draw GK parallel to QP, meeting the section or its opposite again in K, join DK, which is ordinately applied to the same diameter.

Let H be the point in which QP, produced if necessary, meets DK, and since GK is parallel to CH [*constr.*], and CG equal to CD [*by the same*], HK is equal to HD [2. 6 *and Cor.* 13. 5 *Eu.*), and therefore DK is ordinately applied to the diameter QP [32. 1 *Sup*].



Let now the conick section be a parabola; PQ being the given diameter, and D the point, from which the ordinate is required to be drawn.

Draw from D to the given diameter, any right line whatever DQ, on which produced, take QR equal to DQ, through R, draw RK parallel to PQ, meeting the parabola in K, join DK, which is ordinately applied to the diameter PQ.

Let H be the point, in which DK meets PQ, and since HQ is parallel to KR, and DQ equal to QR [*constr.*], DH is equal to

HK [2. 6 *and Cor.* 13. 5 *Eu.*), and therefore **DK is ordinately** applied to the diameter PQ [32. 1 *Sup*].

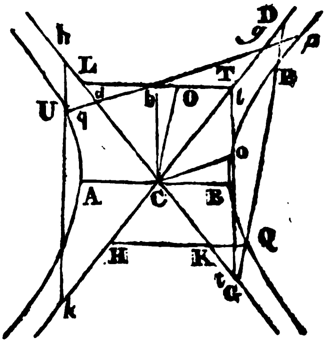
Cor. Hence it appears, how a right line, ordinately applied to a diameter of a conick section, may be drawn from any point within the section, namely, by drawing through any point in the section, a right line ordinately applied to the diameter, by this prop. and drawing through the given point, a right line parallel to the right line so applied.

PROP. XXXVII. THEOR.

If a right line, touching a hyperbola, or cutting a hyperbola or opposite hyperbolas, meet both asymptotes, the square of the segment of the tangent, between the hyperbola and either asymptote, or rectangle under the segments of the secant, between either of the points in which it meets the hyperbola or opposite hyperbolas, and the asymptotes, or between either asymptote, and the hyperbola or opposite hyperbolas, is equal to the square of the semidiameter which is parallel to the tangent or secant.

And the segment of the tangent, between the asymptotes, is bisected in the contact; and the segments of the secant, between the hyperbola or opposite hyperbolas, and the adjacent asymptotes, are equal.

Case 1. If the semidiameter, to which the tangent or secant is parallel, be an axis, the proposition is demonstrated in prop. 25 of this, except the last paragraph, which, however, in this case, is manifest from the proof of that proposition.



Case 2. But let a right line DG parallel to a second semidiameter CO, which is not a semiaxis, meet the hyperbola BP in P and Q, and the asymptotes in D and G; the rectangle DQG or PDQ is equal to the square of CO, and DP is equal to QG.

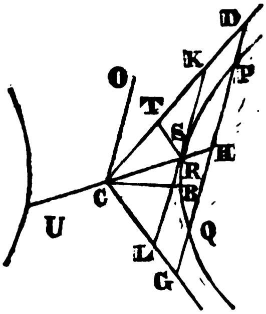
Through Q and O draw HQ and LT parallel to the transverse semiaxis CB, meeting the asymptotes in H and K, L and T.

Because the triangles QDH and OCT are equiangular [29. *and* 32. 1 *Eu.*], DQ is to QH, as CO is to OT [4. 6. *Eu.*); and, because the triangles GQK and COL are equiangular [*Cor.* 3. 9. 1 *Sup.*], GQ is to QK, as CO is to OL [4. 6. *Eu.*], therefore, compounding these two ratios, the rectangle DQG is to the rectangle HQK, as the square of CO is to the rectangle LOT [23. 6. *and* 22. 5 *Eu*] ; but the point O being in one of the hyperbolas conjugate to the hyperbolas AU and BP [*Def.* 7. 1 *Sup.*], the rectangles HQK and LOT are each of them equal to the square of CB [25. 1 *Sup.*], and therefore to each other; therefore the rectangle DQG and the square of CO, having the same ratio to the equal rectangles HQK and LOT, are equal [9. 5 *Eu*].

In like manner, the rectangle DPG may be proved equal to the square of CO, therefore the rectangles DPG and DQG are equal to each other, and so DP equal to QG [*Cor.* 1. 7. 2 *Eu.*], and, adding to each PQ, the right line DQ is equal to PG ; therefore the rectangles QGP, PDQ, DPG and DQG are each of them equal to the square of CO, the segments DP and QG of the secant DG, between the hyperbola BP, and the asymptotes CD and CG being equal.

Case 3. If a right line dg, parallel to a transverse semidiameter Co, which is not a semiaxis, meet the opposite hyperbolas AU and BP in q and p, and the asymptotes in d and g, drawing through q and o, hk and lt parallel to the second semiaxis cb ; it may in like manner be demonstrated, that either of the rectangles dqg, dpg, qdp or qgp is equal to the square of Co, and the segments qd and gp equal, using the same proof, as in the preceding case, only substituting for the large letters O, B, D, P, Q, G, H, K, L and T, the corresponding small ones, and using the expression, in one of the opposite hyperbolas AU or BP, instead of, in one of those conjugate thereto.

Case 4. Let now KL, parallel to the semidiameter CO, touch the hyperbola RP in R, and meet the asymptotes in K and L ; and, in like manner as in case 2, by drawing through R and O, right lines parallel to the principal semiaxis CB, the rectangle KRL may be proved equal to the square of CO : join CR, in which produced take any point H, through which draw DG parallel to KL, meeting the asymptotes in D and G, and the hyperbola in P and Q : PQ is ordinately applied to the diameter CH [*Def.* 12. 1 *Sup.*], and therefore bisected in H [31. 1 *Sup.*]; whence DP being equal to QG [*case* 2 *of this prop.*], DH is equal to HG



[*Ax.* 2. 1 *Eu.*] ; and since, because of the equiangular triangles CRK and CHD, CRL and CHG, KR is to RC, as DH to HC [4. 6 *Eu.*], and RC to RL, as HC to HG [*by the same*], by equality, KR is to RL, as DH is to HG [22. 5 *Eu.*] ; whence, DH being equal to HG, KR is equal to RL [*Cor.* 13. 5 *Eu.*] ; therefore the right line KL is bisected in R, and so, the rectangle KRL being equal to the square of CO, the square of RK is equal to the same square of CO, the segment KL of the tangent, between the asymptotes being bisected in the contact.

Cor. 1. If a right line [KL], terminated by the asymptotes [CK and CL] of a hyperbola, be bisected in a point [R] in which it meets the hyperbola, it touches the section in that point.

For if not, let it, if possible, meet it in any other point as S, SK would be equal to RL [*by this prop.*], or its equal [*Hyp.*], RK, the part to the whole, which is absurd, therefore KL does not meet the hyperbola in any other point but R, and therefore touches it in that point [*Def.* 10. 1 *Sup*].

Cor. 2 The segment [RK], of a tangent to a hyperbola, between the hyperbola and an asymptote, is equal to the semidiameter [CO] parallel to it, and the segment [KL], between the asymptotes, to the diameter.

Cor. 3 If in a right line [qp, see the first fig. of this prop], terminated by opposite hyperbolas, or in a right line [QP] terminated by the same hyperbola produced, two points [d and g, or D and G] be so taken, that the rectangle [qdp and qgp, or QDP and QGP] under the distances of either assumed point from its extremes, be equal to the square of the semidiameter [Co or CO] parallel thereto, for which see Cor. 2 and 3. 6. 2 Eu, the assumed points are in the asymptotes.

Cor. 4. Hence, also, a transverse diameter [UR, see the prec. fig.], is less than any right line parallel to it, and terminated by the opposite hyperbolas.

Cor. 5. If a tangent [KL, see the second fig of this prop.], to a hyperbola, meet both asymptotes, and from the contact [R] a right line [RT] be drawn to one asymptote parallel to the other ; the rectangle KCL] under the segments of the asymptotes between the tangent and the centre, is fourfold the rectangle [CTR], under the right line [RT] so drawn, and the segment [CT] of the asymptote, between that right line and the centre.

For, because of the similar triangles KTR and KCL, and the equality of KR and RL [*by this prop.*], CK is double to CT and CL to TR, therefore the rectangle KCL is similar to the rectangle CTR, and is therefore to it in a duplicate ratio of CK to CT (20. 6 *Eu.*); whence CK being double to CT, the rectangle KCL is fourfold the rectangle CTR.

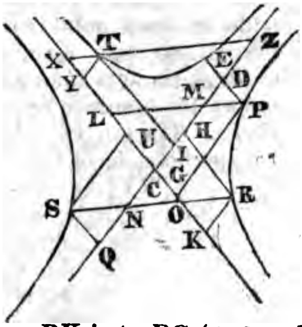
Cor. 6. From a given point (K), in an asymptote (CK), to draw a tangent to an adjacent hyperbola (RP).

Find the centre C of the hyperbola (33. 1 *Sup.*), and having drawn the asymptote CL (35. 1 *and Def.* 19. 1 *Sup.*), bisect CK in T, draw TR parallel to CL, meeting the hyperbola in R, and draw KRL meeting the other asymptote CG in L; and, because CT is equal to TK and TR parallel to CL (*constr.*), LR is equal to RK (2. 6 *and Cor.* 13. 5 *Eu.*), and therefore KRL touches the hyperbola in R (*Cor.* 1 *to this prop*).

PROP. XXXVIII. THEOR.

If from any point in a hyperbola, two right lines be drawn to the asymptotes, and from any other point in the same, or an opposite or conjugate hyperbola, there be drawn to the asymptotes two other right lines parallel to the former; the rectangle under the two first drawn right lines, is equal to the rectangle under the other two.

Case. 1. Let PD and PG be two right lines, drawn from a point P in a hyperbola to the asymptotes CD and CK; and RH and RK parallel to PD and PG be drawn to the same asymptotes from a point R in the same hyperbola; the rectangles DPG and HRK are equal.



Through P and R draw PL and RN parallel to each other meeting the asymptotes in M and L, O and N; the rectangles MPL and ORN are equal (37. 1 *Sup. and Ax.* 1. 1 *Eu.*), and, because the triangles PDM and RHN are equiangular (*Cor.* 3. 9. 1 *Sup.*), PD is to HR, as MP is to NR (4. 6 *and* 16. 5 *Eu.*), or, (because of the equal rectangles MPL and ORN), as OR is to LP (16. 6 *Eu.*); or (because of the equiangular triangles OKR and LGP), as RK is to PG (4. 6 *and* 16. 5 *Eu.*); therefore the rectangles DPG and HRK are equal (16. 6 *Eu*).

Case 2. In like manner, if the right lines SQ and SU parallel to PD and PG be drawn to the asymptotes, from a point S, in the hyperbola opposite to PR,; through P and S draw PL and SO parallel to each other, meeting the asymptotes in L and M, N and O; and, because of the equiangular triangles PDM and SQN, PD is to SQ, as PM to SN (4. 6 *and* 16. 5 *Eu.*), or, because of the equal rectangles MPL and NSO (37. 1 *Sup. and Ax.* 1. 1 *Eu.*), as SO to LP (16. 6 *Eu.*), or, because of the equiangular triangles SUO and PGL, as SU to PG (4. 6 *and* 16. 5 *Eu.*), therefore the rectangles DPG and QSU are equal (16. 6 *Eu*).

Case 3. Let now the right lines TI and TY, parallel to PD and PG, be drawn to the asymptotes, from a point T, in a hyperbola conjugate to PR; through P and T draw PL and XZ parallel to each other, meeting the asymptotes in L and M, X and Z; and, because of the equiangular triangles PDM and XYT, PD is to XY, as PM to XT (4. 6 *and* 16. 5 *Eu.*), or, because the rectangles MPL and XTZ are equal (37. 1 *Sup. and Ax.* 1. 1. *Eu.*), as TZ to LP (16. 6 *Eu.*), or, because of the equiangular triangles TIZ and LGP, as ZI to PG (4. 6 *and* 16. 5 *Eu.*), therefore the rectangle DPG is equal to the rectangle under XY and IZ (16. 6 *Eu.*); but, because of the equiangular triangles XYT and TIZ, XY is to YT, as TI to IZ, and therefore the rectangle YTI is equal to the rectangle under XY and IZ (16. 6 *Eu.*); whence the rectangles DPG and YTI, being each equal to that under XY and IZ, are equal to each other.

Cor. 1. Since right lines drawn from a given point to a given right line in equal angles are equal, as is manifest from 5. 1 *Eu*, it follows from this proposition, that the rectangle under right lines, drawn from any point, in four conjugate hyperbolas, to the asymptotes, in given angles, is of the same magnitude, from what point so ever of the hyperbolas it be drawn.

Cor. 2. If from any two points (P and T), in four conjugate hyperbolas, right lines (PD and TY) be drawn to one asymptote, parallel to the other; the rectangles (PDC and TYC) under the right lines so drawn, and the segments of the asymptotes between them and the centre, are equal.

Draw PG parallel to CD and TI to CY; and because of the parallelograms CT and CP, the right lines CG, CD, CI and CY are severally equal to PD, PG, TY and TI; whence, the rectangles DPG and YTI being equal (*by this prop.*), the rectangles PDC and TYC are also equal.

Cor. 3. A right line (EP), terminated by adjacent hyperbolas, and parallel to one asymptote (CK), is bisected by the other (CD).

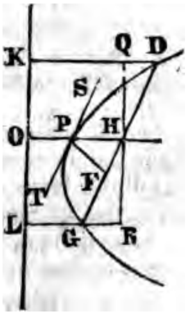
For the rectangles PDC and EDC are equal (*prec. cor.*); whence, the side CD being common to both rectangles, ED is equal to DP.

Cor. 4. The rectangle, under the segments of the asymptotes, cut off by a tangent to any of four conjugate hyperbolas, towards the centre, is equal to the rectangle, under the segments of the asymptotes so cut off by any other tangent.

For these rectangles, are fourfold the rectangles, under right lines, drawn from each of the contacts to one asymptote, parallel to the other, and the segments of the asymptotes between the right lines so drawn and the centre (*Cor.* 5. 37. 1 *Sup.*), which latter rectangles are (*by cor.* 2 *of this prop.*) equal.

PROP. XXXIX. THEOR.

A right line (DG), terminated by a parabola, and passing through its focus (F), is equal to the parameter, of the diameter (PH), to which it is ordinately applied.



Let P be the vertex of the diameter PH, and KL the directrix, which let HP produced meet in O, draw DK and GL at right angles to KL, join FP, through the point H in which DG meets PH, draw the right line QHR at right angles to OH, meeting KD and LG in Q and R, and bisect the angle OPF by the right line ST, which is a tangent to the parabola (10. 1 *Sup.*), and parallel to DG (*Def.* 12. 1 *Sup.*), therefore the alternate angles PFH and FPT are equal, and the external OPT to the internal remote PHF (29. 1 *Eu.*); whence the angles PFH and PHF being severally equal to the equals FPT and OPT, are equal to each other, and so PH is equal to PF (6. 1 *Eu.*), or its equal (*Def.* 8. 1 *Sup.*), PO; and QR is at right angles to OH (*constr.*), it is therefore at right angles to DK and LR (28 *and* 29. 1 *Eu.*); whence, the triangles HDQ and HGR, being right angled at Q and R, and having the angles at H equal (15. 1 *Eu.*), are equiangular, and so, having also DH equal to HG (31. 1 *Sup.*), QD is equal to GR (26. 1 *Eu.*); therefore DK and GL together are equal KQ and LR together, or to twice OH;

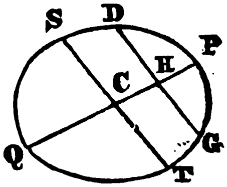
whence FD being equal to DK, and FG to GL (*Def.* 8. 1 *Sup.*), DG is double of OH, and fourfold of OP or PF, and therefore equal to the parameter of the diameter PH (*Def.* 16. 1 *Sup*).

Cor. The segment (OH), of a diameter of a parabola, intercepted between the directrix, and the ordinate (GD) which passes through the focus, is bisected in its vertex.

PROP. XL. THEOR.

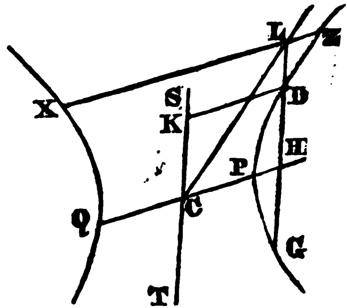
If from any point of a conick section, an ordinate be drawn to a diameter; the square of the ordinate is, in the case of an ellipse or transverse diameter of a hyperbola, to the rectangle under the abscissas, and in the case of the second diameter of a hyperbola, to the sum of the squares of the second semidiameter, and the segment thereof between the centre and ordinate, as the square of the semidiameter which is parallel to the ordinate, to the square of that which it meets; and, in the case of a parabola, the square of the ordinate, is equal to the rectangle, under the abscissa, and the parameter of the diameter, to which the ordinate is drawn.

Part 1. Let DH be an ordinate, drawn from any point D of an ellipse, to a diameter QP, and let ST be the diameter parallel to DH, C being the centre. The square of DH is to the rectangle QHP, as the square of SC is to the square of CP.



Let DH produced meet the ellipse again in G; the rectangle DHG is to the rectangle QHP, as the rectangle SCT is to the rectangle QCP (14. 1 *Sup.*); but DG is bisected in H (31. 1 *Sup.*), and QP and ST are bisected in C (5. 1 *Sup.*); therefore the square of DH is the rectangle QHP, as the square of SC is to the square of CP.

Part 2. Let now DH be an ordinate, drawn from any point D of a hyperbola, to a transverse diameter QPH, C being the centre, and SC the second semidiameter to which DH is parallel. The square of DH is to the rectangle QHP, as the square of SC is to the square of CP.

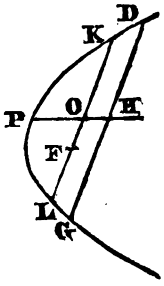


Produce GD to meet the asymptote CL in L, and through L, draw XLZ parallel to CP, meeting the opposite hyperbolas in X and Z; the rectangle DHG is to the rectangle QHP, as the rectangle DLG is to the rectangle XLZ (14. 1 *Sup.*); but the rectangle DLG is equal to the square of CS (37. 1 *Sup.*), and the rectangle XLZ is equal to the square of CP (*by the same*); therefore the rectangle DHG, or, DG being bisected in H (31. 1 *Sup.*), the square of DH, is to the rectangle QHP, as the square of CS is to the square of CP.

Part 3. Let an ordinate DK meet a second diameter TS. The square of DK is to the squares of CS and CK together, as the square of CP is to the square of CS.

For (*by the preceding part and inverting*), the rectangle QHP is to the square of DH or CK, as the square of CP is to the square of CS; therefore, the rectangle QHP with the square of CP, or, which is equal (6. 2 *Eu.*), the square of CH or DK, is to the squares of CS and CK together, as the square of CP is to the square of CS (12. 5 *Eu*).

Part 4. Lastly, let DH be an ordinate to a diameter PH of a parabola. The square of DH, is equal to a rectangle, under the abscissa PH, and the parameter of the diameter PH.



Find the focus F (35. 1 *Sup.*), through which, draw KL parallel to DG, meeting the parabola in K and L, and PH in O; KL is ordinately applied to the diameter PH (*Constr. and Def.* 12. 1 *Sup.*), and therefore equal to the parameter of that diameter (39. 1 *Sup*), and therefore fourfold of PO (*Def.* 16. 1 *Sup. and Cor.* 39. 1 *Sup.*), whence, OK, being the half of KL (31. 1 *Sup.*), is double to PO, and so PO, OK and KL are continually proportional (*Theor.* 1. 15. 5 *Eu.*), therefore the square of KO is equal to the rectangle under PO and KL (17. 6 *Eu.*); but the square of KO is to the square of DH, as PO is to PH (20 *and* 31. 1 *Sup.*), or, (1. 6 *Eu.*), as the rectangle under PO and KL is to the rectangle under PH and KL; whence, the square of KO having been just proved equal to the rectangle under PO and KL, the square of DH is equal to the rectangle under PH and KL (14. 5 *Eu.*), or, KL being equal to the parameter of the diameter PH (39. 1 *Sup.*), to the rectangle under PH and that parameter.

Cor. 1. If from two points of an ellipse, hyperbola or opposite hyperbolas, ordinates be drawn to the same diameter; the ratios of the squares of the ordinates, in the case of an ellipse or transverse diameter of a hyperbola, to the rectangles under their respective abscissas, and, in the case of a second diameter of a hyperbola, to the sums of the squares of the second semidiameter and the segments thereof between the centre and ordinates, are equal, being each, by this proposition, equal to the ratio of the square of the semidiameter which is parallel to the ordinates, to the square of that which they meet.

Cor. 2. And if from two points of a conick section or opposite sections, ordinates be drawn to the same semidiameter; the ratio of the squares of the ordinates to each other is, in the case of an ellipse or transverse diameter of a hyperbola, equal to that of the rectangles under their respective abscissas, in the case of a second diameter of a hyperbola, to that of the sums of the squares of the semidiameter and the segments thereof between the centre and ordinates, and, in the case of a parabola, to that of the abscissas; the two former cases being manifest from the prec. cor. and 16. 5 Eu, and the last from this prop. and 1. 6 Eu.

Cor. 3. If from any point of a conick section, an ordinate be drawn to a diameter which is not the second diameter of a hyperbola, and from any point of the diameter within the section or opposite sections, a right line be drawn parallel to the ordinate, of which the square is to the square of the ordinate, in the case of an ellipse or hyperbola, as the rectangles under the segments of the diameter between the parallels and its vertices, and, in the case of a parabola, as the segments of the diameter, between the parallels and its vertex, the extreme of the right line drawn parallel to the ordinate, which is remote from the diameter, is in the section, within which the point in the diameter is taken.

For if the right line drawn parallel to the ordinate met the section in any other point, its square would not be to the square of the ordinate, in the ratio demonstrated in the preceding corollary.

Cor. 4. If to any diameter of an ellipse or hyperbola, an ordinate be drawn; the rectangle under the abscissas, in the case of an ellipse or transverse diameter of hyperbola, and the sum of the squares of the semidiameter and the segment thereof between the centre and ordinate, in the case of a second diameter of a hyperbola, is to the square of the ordinate, as the diameter is to its parameter, as is manifest from this proposition, Def. 15. 1. Sup. and Cor. 2. 20. 6 Eu.

Fig. 1.

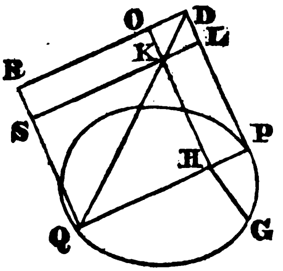
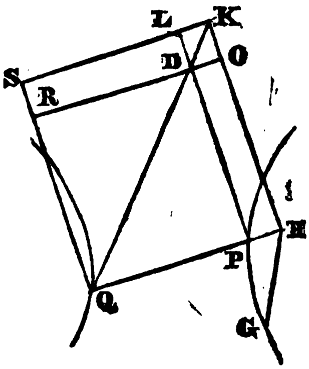


Fig. 2.



Scholium. Let GH, see fig. 1 and 2, be an ordinate to any diameter of an ellipse or a transverse one of a hyperbola, let that diameter be QP, and from its vertex P draw PD at right angles to QP equal to its parameter; complete the parallelogram PR, join QD, and draw HKO parallel to PD, meeting QD and RD in K and O, and through K, SKL parallel to QP, meeting QR and PD, produced if necessary, in S and L.

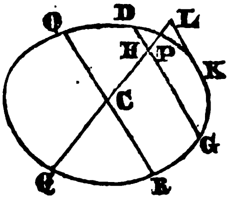
In the case of the ellipse, fig. 1, the rectangle QHP is to the square of HG, as the diameter QP to its parameter PD (*Cor.* 4. *to this prop.*), or, because of the parallels, as QH is to HK, or (1. 6 *Eu.*), as the same rectangle QHP to the rectangle KHP; therefore the square of HG is equal to the rectangle KHP (9. 5 *Eu.*), and, of course, being applied to the parameter PD with the attitude of the abscissa PH adjacent to PD, is deficient by a figure OL similar to the rectangle RP under the diameter QP and its parameter PD (*Def.* 5. 6 *Eu*). A similar consequence would follow, if the square of GH were applied to the parameter QR, from the point Q, with the attitude QH. On account of this deficiency, Appolonius named this line, an *Ellipse* or *Ellipsis*, which, in the Greek language, signifies a deficiency.

In like manner, in the case of the hyperbola, fig. 2, the rectangle QHP is to the square of GH, as the diameter QP to its parameter PD (*Cor.* 4 *to this prop.*), or, because of the parallels, as QH to HK, or (1. 6 *Eu.*), as the same rectangle QHP to the rectangle KHP; therefore the square of GH is equal to the rectangle KHP (9. 5 *Eu.*), and of course being applied to the parameter PD with the attitude of the less abscissa HP, exceeds by a figure LO similar to the rectangle RP under the diameter QP and its parameter PD (*Def.* 6. 6 *En*). On account of this excess, Apollonius named this line, a *Hyperbola* or *Hyperbole*, which, in the Greek language, signifies a redundancy.

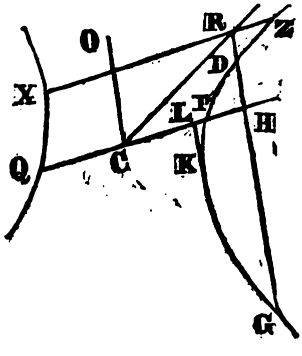
And the square of an ordinate to any diameter of a parabola is, by this prop. equal to the rectangle under the abscissa and the parameter of the diameter. For this reason, Apollonius named this line, a *Parabola* or *Parabole*, which, in the Greek language, signifies similitude or equality.

PROP. XLI. THEOR.

If a right line, touching a conick section, or cutting in two points a conick section or opposite sections, meet any diameter; the square of the segment of the tangent, or rectangle under the segments of the secant, between the diameter, and the point or points in which it meets the section or sections, is, in the case of an ellipse or transverse diameter of a hyperbola, to the rectangle under the segments of the diameter, between the tangent or secant and its vertices, and, in the case of a second diameter of a hyperbola, to the sum of the squares of the semidiameter, and the segment thereof between the centre, and the tangent or secant, as the square of the semidiameter to which the tangent or secant is parallel, to the square of the semidiameter which it meets; and, in the case of a parabola, is equal to the rectangle under the segment of the diameter, between its vertex, and the tangent or secant, and the parameter of the diameter, whose ordinates are parallel to the tangent or secant.



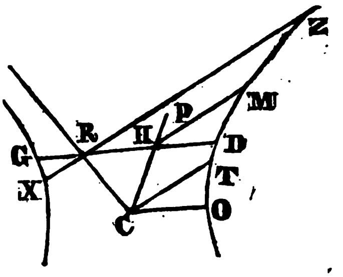
First, let the figure be an ellipse; let KL be a tangent, touching it in K, and meeting a diameter QP in L, let OCR be a diameter parallel to KL, C being the centre, and let DG be a secant, parallel to the diameter OR, meeting the ellipse in D and G and the diameter QP in H; the square of KL is to the rectangle PLQ, and the rectangle DHG to the rectangle QHP, as the rectangle OCR is to the rectangle OCP (14. 1 *Sup.*), or, OR and QP being bisected in C (5. 1 *Sup.*), as the square of CO is to the square of CP.



Secondly, let the tangent KL, or secant DHG, meet a transverse diameter QP, of a hyperbola PD, in L or H, CO being the semidiameter parallel to the tangent or secant; the square of KL is to the rectangle QLP, and the rectangle DHG to the rectangle QHP, as the square of CO is to the square of CP.

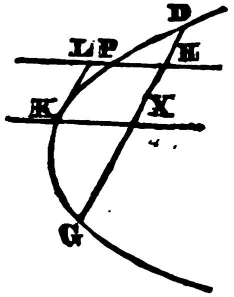
Draw the asymptote CR (35. 1 *and Def.* 19. 1 *Sup.*), which let GD produced meet in R, and through R draw XRZ parallel to CP, meeting the opposite hyperbolas in X and Z; the rectangle DHG is to the rectangle QHP, as the rectangle DRG is to the rectangle XRZ (14. 1 *Sup.*); but the rectangle DRG is equal to the square of CO (37. 1 *Sup.*), and the rectangle XRZ to the square of CP (*by the same*), therefore the rectangle DHG is to the rectangle QHP, as the square of CO is to the square of CP.

And since the square of KL is to the rectangle QLP, as the rectangle DHG is to the rectangle QHP (14. 1 *Sup.*), and it has been just proved, that the rectangle DHG is to the rectangle QHP, as the square of CO is to the square of CP, therefore the square of CL is to the rectangle QLP as the square of CO is to the square of CP.



Thirdly, let a right line GHD meet opposite hyperbolas in G and D, and a second diameter CP in H, CO being the semidiameter parallel to GD; the rectangle GHD is to the squares of CP and CH together, as the square of CO is to the square of CP.

From H draw the ordinate HM to the diameter CP (*Cor.* 36. 1 *Sup.*), and let CT be the semidiameter parallel to HM, let GD meet the asymptote CR in R, and through R, draw XRZ parallel to HM or CT, meeting the opposite hyperbolas in X and Z; the rectangle GHD is to the square of HM, as the rectangle GRD is to the rectangle XRZ (14 *and* 31. 1 *Sup.*), or, the rectangle GRD being equal to the square of CO (37. 1 *Sup.*), and the rectangle XRZ to the square of CT (*by the same*), as the square of CO is to the square of CT; and the square of HM is to the squares of CP and CH together, as the square of CT is to the square of CP (40. 1 *Sup.*); therefore, by equality, the rectangle GHD is to the squares of CP and CH together, as the square of CO is to the square of CP (22. 5 *Eu*).



Fourthly, let a right line KL, touching a parabola in K, meet a diameter LPH in L, or DG parallel to KL, meet the parabola in D and G, and the diameter in H, and let KX be the diameter passing through the point K, and to which of course DG is ordinately applied (*Def.* 12. 1 *Sup.*); the square of KL is equal to the rectangle under LP and the parameter of the diameter KX, and the rectangle DHG to the rectangle under PH and the same parameter.

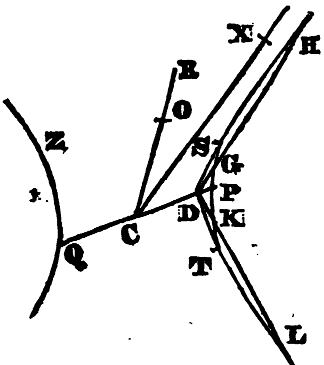
For since DG is ordinately applied to the diameter KX, *it* is bisected in X (31. 1 *Sup*), therefore the square of DX is equal to the rectangle DXG, and is therefore to the rectangle DHG, as KX is to PH (21. 1 *Sup.*), or (1. 6 *Eu.*), as the rectangle under KX and the parameter of the diameter KX is to the rectangle under PH and the same parameter; but the square of DX is equal to the rectangle under KX and the parameter of the diameter KX (40. 1 *Sup.*), therefore the rectangle DHG is equal to the rectangle under PH and the same parameter (14. 5 *Eu*).

And the rectangle DHG is to the square of KL, as PH is to LP (20. 1 *Sup.*), or (1. 6 *Eu.*), as the rectangle under PH and the parameter of the diameter KX is the rectangle under LP and the same parameter; whence the rectangle DHG having been just proved equal to the rectangle under PH and that parameter, the square of KL is equal to the rectangle under LP and the same parameter (14. 5 *Eu*).

PROP. XLII. THEOR.

If two right lines meeting each other, both touch or both cut in two points, or one of them touch, and the other so cut, a conick section or opposite sections; the squares of the segments of the tangents, or rectangles under the segments of the secants, between the concourse of the right lines, and the points wherein they meet the section or sections, are to each other, in the case of an ellipse or hyperbola, as the squares of the semidiameters to which they are parallel; and, in that of a parabola, as the parameters of the diameters, whose ordinates are parallel to the tangents or secants.

Case 1. If the figure be an ellipse, or the right lines meeting each other, be parallel to transverse diameters of a hyperbola, the proposition is manifest from the 14th and 5th propositions of this book.



Case 2. If not, let the concourse D of the right lines meeting each other and a hyperbola or opposite hyperbolas, be in a transverse diameter QP, DS and DT being tangents, and DGH and DKL secants meeting in the point D.

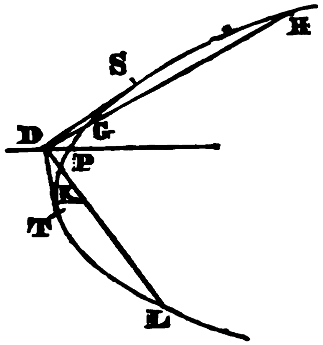
Since the squares of DS and DT, or the rectangles GDH and KDL, have to the rectangle QDP the same ratio, as the squares of the semidiameters to which they are respectively parallel, to the square of the semidiameter CP (41. 1 *Sup.*), by inverting and ordinate equality, these squares or rectangles are to each other, as the squares of the semidiameters to which they are respectively parallel (*Theor.* 3. 15. 5 *and* 22. 5 *Eu*).

Case 3. When the concourse of two right lines, meeting a hyperbola, as PS, or opposite hyperbolas PS and QZ, is in a second diameter, as at R, CO being the second diameter.

Since the squares of the segments of the tangents or rectangles under the segments of the secants between the point R and the hyperbola, or hyperbolas, have the same ratio to the sum of the squares of CO and CR, as the squares of the semidiameters to which they are respectively parallel, have to the square of CO (41. 1 *Sup.*); by inverting and ordinate equality, these squares or rectangles are to each other, as the squares of the semidiameters to which they are respectively parallel (*Theor.* 3. 15. 5 *and* 22. 5. *Eu*).

Case 4. When the concourse is an asymptote CX of a hyperbola, as at X.

Since the square or rectangles are equal to the squares of the semidiameters to which they are parallel (37. 1 *Sup.*), they are to each other as the same squares (*Schol.* 7. 5 *Eu*).



Case 5. When the figure is a parabola, D being the concourse of the tangents or secants.

Through D, draw the diameter DP, of which let P be the vertex; and since the squares of the segments of the tangents, or rectangles under the segments of the secants between D and the section, are equal to the rectangles under DP and the parameters of the diameters, to which right lines parallel to the tangents or secants are ordinately applied (41. 1 *Sup.*), these squares or rectangles are to each other as the same parameters (1. 6 *and* 11. 5 *Eu*).

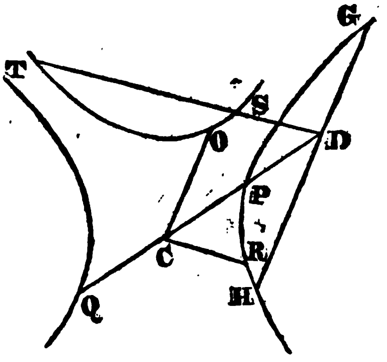
Scholium. Since the rectangle under any diameter of an ellipse or hyperbola and its parameter, is equal to the square of its conjugate (*Def.* 15. 1 *Sup. and* 17. 6 *Eu.*), it follows, that the squares of the segments of the tangents, or rectangles under the segments of the secants, mentioned in this proposition, are to each other, in ellipses and hyperbolas, in a ratio, compounded of the ratios of the diameters, whose ordinates are parallel to the tangents or secants, and their parameters (23. 6 *Eu.*); being, in parabolas, when the diameters are infinite, and therefore to each other in a ratio of equality, in the simple ratio of these parameters. Whence appears a further analogy between the different sections.

PROP. XLIII. THEOR.

If one, of two right lines meeting each other, touch or cut in two points, a hyperbola or opposite hyperbolas, and the other so touch or cut a hyperbola or opposite hyperbolas conjugate to the former; the square of the segment of the former tangent, or rectangle under the segments of the former secant, between the concourse and the section or sections, is to the square of, or rectangle under, the like segment or segments of the other tangent or secant, when the concourse is in an asymptote, as the squares of the semidiameters, which are parallel to the right lines so meeting each other; and when the concourse is not in an asymptote, the square or rectangle pertaining to the hyperbola or hyperbolas, to which the diameter passing the concourse of the meeting right lines is a transverse one, is to the square or rectangle pertaining to the conjugate hyperbola or hyperbolas, in a ratio compounded of the ratios of the semidiameters to which the meeting right lines are parallel, and of the ratio of the rectangle under the distances of the concourse from the vertices of the diameter passing through it, to the sum of the squares of the semidiameter which passes through the concourse, and the distance of the concourse from the centre.

Case 1. If the concourse of the right lines so meeting each other be in an asymptote, these squares or rectangles are equal to the squares of the semidiameters to which the same right lines are parallel (37. 1 *Sup.*), and are therefore to each other, as the squares of the same semidiameters (*Schol* 7. 5 *Eu*).

Case 2. Let GH and TD be two right lines meeting each other in D, and let one of them cut the hyperbola PG in G and H, and the other TD the hyperbola OT conjugate to PG in S and T; let CO and CR be semidiameters parallel to GH and TD, and QPD the diameter passing through the concourse D; the rectangle GDH is to the rectangle TDS, in a ratio, compounded of the ratios, of the square of CO to the square of CR, and and of the rectangle QDP to the sum of the squares of CP and CD.



For the rectangle GDH is to the rectangle QDP, as the square of CO is to the square of CP (41. 1 *Sup.*), and the rectangle SDT is to the sum of the squares of CP and CD, as the square of CR is to the square of CP (by the same); therefore the rectangle GDH is to the rectangle SDT, in a ratio, compounded of the ratios, of the square of CO to the square of CR, and of the rectangle QDP to the sum of the squares of CP and CD (*Cor.* 6. 23. 6 *Eu*).

The demonstration is similar, if one or both of the right lines meeting each other, were tangents, or cut opposite hyperbolas.

PROP. XLIV. THEOR.

If from any point of a conick section, an ordinate be drawn to a diameter, and a tangent meeting the same diameter; the semidiameter is, in the case of an ellipse or hyperbola, a mean proportional between the segments of the diameter, between the centre and ordinate, and between the centre and tangent; and, in the case of a parabola, the segment of the diameter, between the ordinate and tangent, is bisected in its vertex.

Let PH, see fig. 1, 2 and 3, of this prop. be an ordinate, drawn from any point P of a conick section, to a diameter HK, which is not a second diameter of a hyperbola, or in the case of a hyperbola, fig. 2, let PT be an ordinate drawn to a second diameter CT, and let PK be a tangent drawn from P, meeting the diameter HK in K, and in fig. 2, the diameter CT in X, CG being in fig. 1 and 2, the semidiameter passing through H and K, and CO in fig. 1, that which passes through T and X; CG is, in fig. 1 and 2, a mean proportional between CH and CK; CO, in fig. 2, between CT and CX; and in fig. 3, KH is bisected in G.

Case 1. When the ordinate PH (see fig. 1 and 2), and tangent PK, meet any diameter of an ellipse, or a transverse one of a hyperbola.

Let D and G be the vertices of that diameter, through which, draw DR and GL parallel to PH, meeting PK in R and L, these touch the section in D and G (*Def.* 12. 1 *Sup.*); on CD, produced in fig. 2, take CS equal to CH, DS is equal to GH (*Ax.* 2 and 3. 1 *Eu*).

And because the tangent RPK meets the tangents DR and GL, the square of RP is to the square of PL, as the square of DR is

Fig. 1.

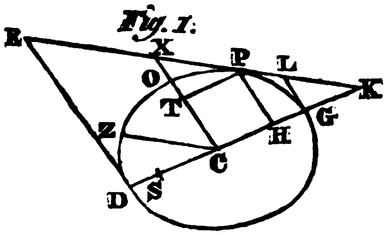
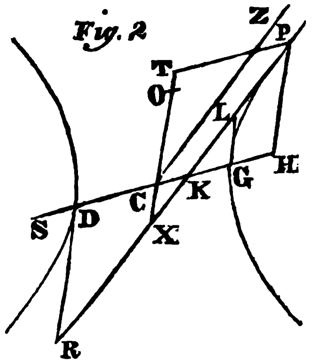


Fig. 2.

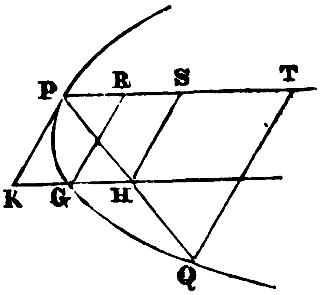


to the square of GL (*Cor.* 2. 14. 1 *Sup.*), therefore RP is to PL, as DR is to GL (22. 6 *Eu.*), or, because of the equiangular triangles RDK and LGK. as DK is to GK (4. 6 and 16. 5 *Eu.*); but, because of the parallels RD, PH and LG, DH is to HG, as RP is to PL (*Cor.* 2. 10. 6 *Eu.*), therefore DH is to HG, as DK is to GK (11. 5 *Eu.*); therefore, by dividing in fig. 1, and compounding in fig. 2, SH is to HG, as DG is to GK (17 and 18 5 *Eu.*); and taking the halves of the antecedents, CH is to HG, as CG is to GK (*Theor.* 1. 15. 5 and 22. 5 *Eu.*), and, by converting, CH is to CG, as CG is to CK.

Case 2. When the ordinate PT (see fig. 2), and tangent PX, meet a second diameter CT of a hyperbola.

Let O be a vertex of the diameter CT, and DG the transverse diameter conjugate to the diameter CT, meeting the tangent PX in K; draw PH an ordinate to the diameter DGH (36. 1 *Sup*).

By the preceding case, CH, CG and CK are continually proportional, therefore the square of CH is to the square of CG, as CH is to CK (*Cor.* 2. 20. 6 *Eu.*), and, by dividing, the excess of the square of CH above that of CG, or, which is equal (6. 2 *Eu.*), the rectangle DHG, is to the square of CG, as KH is to CK (17. 5 *Eu.*); whence, the square of PH being to the square of CO, as the rectangle DHG is to the square of CG (40. 1 *Sup.* and 16. 5 *Eu.*), the ratios of the square of PH to the square of CO, and of KH to CK, being each equal to the ratio of the rectangle DHG to the square of CG, are equal to each other (11. 5 *Eu.*); but, because of the equiangular triangles HKP and CKX, PH is to CX, as KH is to CK (4. 6 and 16. 5 *Eu.*), therefore the square of PH or CT is to the square of CO, as PH or CT is to CX (11. 5 *Eu.*); therefore CT, CO and CX are continually proportional (20. 6 *Eu*).



Case 3. When the ordinate PH and tangent PK meet a diameter GH of a parabola.

Let the ordinate PH be produced to meet the parabola again in Q, and let GR, HS and QT, drawn parallel to PK, meet the diameter PT drawn through P; GR and QT are ordinates to the diameter PT (*Def.* 12. 1 *Sup*).

And since PQ is double to PH, (31. 1 *Sup.*), PT is double to PS (2. 6 *Eu.*), and QT to HS (4. 6 and 16. 5 *Eu.*), or GR, therefore the square of QT is fourfold the square of GR (*Cor.* 4. 2 *Eu.*), and therefore the abscissa PT fourfold the abscissa PR (*Cor.* 2. 40. 1 *Sup*), and PS or KH double to PR or KG, and so KH is bisected in G.

Cor. If from the vertices (P and G, see fig 3), of two diameters (PT and GH) of a parabola, ordinates (PH and GR) be drawn to the same diameters ; the abscissas (GH and PR) are equal.

For, the tangent PK being drawn from P, meeting GH in K, because of the parallelogram PKGR, PR is equal to KG (34. 1 *Eu.*), or, which is equal (*by this prop.*), to GH.

PROP. XLV. THEOR.

If from any point of an ellipse or hyperbola, an ordinate be drawn to any diameter, and a tangent from the same point, meet the same diameter ; the rectangle under the segments of the diameter, between the ordinate and centre, and between the ordinate and tangent, is, in the case of an ellipse, or transverse diameter of a hyperbola, equal to the rectangle under the segments of the same diameter, between the ordinate and its vertices ; and, in the case of a second diameter of a hyperbola, to the sum of the squares of the second semidiameter, and segment of the same diameter, between the centre and ordinate.

Let PH, see fig. 1 and 2 of the prec. prop. be an ordinate drawn from any point P of a conick section, to any diameter of an ellipse, or a transverse one of a hyperbola, let this diameter be DG, let PT, see fig. 2, be an ordinate drawn from P to a second diameter CT of a hyperbola, and let a tangent PK drawn from P, in fig. 1, meet DG produced in K, and, in fig. 2, meet

DG in K, and CT in X; the rectangle CHK is, in both figures, equal to the rectangle DHG, and, in fig. 2, the rectangle CTX is equal to the sum of the squares of CO and CT.

Part 1. The rectangle CHK in fig. 1 and 2, is equal to the rectangle DHG.

For the rectangle HCK is equal to the square of CG (44. 1 *Sup.* and 17. 6 *Eu.*), therefore, in the ellipse, fig. 1, taking from each the square of CH, the excess of the rectangle HCK above the square of CH, or, (3. 2 *Eu.*), the rectangle CHK, is equal to the excess of the square of CG above that of CH, or (5. 2 *Eu.*). the rectangle DHG; and, in the hyperbola, fig. 2, taking these equals from the square of CH, the excess of the square of CH above the rectangle HCK, or (2. 2 *Eu.*), the rectangle CHK, is equal to the excess of the square of CH above that of CG, or (6. 2 *Eu.*), the rectangle DHG.

Part 2. In fig. 2, the rectangle CTX is equal to the sum of the squares of CO and CT.

For the rectangle TCX is equal to the square of CO (44. 1 *Sup.* and 17. 6 *Eu.*), adding to each the square of CT, the rectangle TCX with the square of CT, or (3. 2 *Eu.*), the rectangle CTX is equal to the sum of the squares of CO and CT.

PROP. XLVI. THEOR.

The same things being supposed; the rectangle under the segments of the diameter, between the tangent and centre, and between the tangent and ordinate, is, in the case of an ellipse, or transverse diameter of a hyperbola, equal to the rectangle under the segments of the same, between the tangent and its vertices; and, in the case of a second diameter of a hyperbola, to the sum of the squares of the second semidiameter, and the segment of the same diameter, between the centre and tangent.

Part. 1. In fig. 1 and 2 of the 44th prop. the rectangle CKH is equal to the rectangle DKG.

For the rectangle HCK is equal to the square of CG (44. 1 *Sup.* and 17. 6 *Eu.*), therefore, in the ellipse, fig. 1, taking each from the square of CK, the excess of the square of CK above the rectangle HCK, or (2. 2. *Eu.*), the rectangle CKH, is equal to the excess of the square of CK above that of CG, or (6. 2 *Eu.*), the rectangle DKG; and, in the hyperbola, fig. 2, taking from these equals, the square of CK, the excess of the rectangle HCK above the square of CK, or (3. 2 *Eu.*), the rectangle CKH, is equal to the excess of the square of CG above that of CK, or (5. 2 *Eu.*), the rectangle DKG.

Part 2. In fig. 2, the rectangle CXT is equal to the sum of the squares of CO and CX.

For the rectangle TCX is equal to the square of CO (44. 1 *Sup. and* 17. 6 *Eu.*), adding to each the square of CX, the rectangle TCX with the square of CX, or (3. 2 *Eu.*), the rectangle CXT, is equal to the sum of the squares of CO and CX.

PROP. XLVII. THEOR.

If two parallel right lines (DR and GL, see fig. 1 *and* 2 *to prop.* 44*), touching an ellipse or opposite hyperbolas, meet another tangent (RLK); the rectangle under the segments (DR and GL) of the parallels, between their contacts and the tangent which they meet, is equal to the square of the semidiameter (CO), to which they are parallel. And the rectangle (RPL), under the segments of the tangent (RL), which the parallels meet, between its contact (P), and the parallel tangents, is equal to the square of the semidiameter (CZ), which is parallel to it; as is the rectangle (XPK), under the segments of any tangent (RP), meeting two conjugate diameters (CO and DG), between the contact (P) and the diameters.*

Part 1. The rectangle under DR and GL is equal to the square of CO.

The right line DG joining the contacts D and G is a diameter, for if G were not the other vertex of the diameter passing through D, a right line drawn from G parallel to DR, to the diameter passing through D, would meet that diameter within the section, for it is parallel to the tangent drawn through the vertex of the diameter remote from D (30. 1 *Sup.* and 30. 1 *Eu.*), which tangent falling wholly without the section (*Def.* 10. 1 *Sup.*), if the right line so drawn from G, did not meet that diameter within the section, it would meet the tangent drawn through the vertex remote from D, contrary to the definition of parallel right lines; therefore if DG were not a diameter, a right line drawn through G parallel to DR would enter the section and not be a tangent, contrary to the supposition.

And the diameter CO is conjugate to DG (*Def.* 14. 1 *Sup.*), and, if, in the ellipse, RL be parallel to DG, the proposition, as far as relates to the rectangle under DR and GL and the rectangle RPL is manifest.

But if RL be not parallel to DG in fig. 1, let RL, in fig. 1 and 2, meet DG in K, and CO produced in X, and let ordinates PH and PT be drawn to the diameters DG and CO (36. 1 *Sup*).

And because the rectangle DKG is equal to the rectangle CKH (46. 1 *Sup.*), DK is to CK, as HK is to GK (16. 6 *Eu.*), therefore, because of the parallels, DR is to CX, as HP to GL (4. 6 and 16. 5 *Eu.*), and therefore the rectangle under DR and GL is equal to the rectangle under CX and HP or CT (16. 6 *Eu.*), or, which is equal (44. 1 *Sup. and* 17. 6 *Eu.*), the square of CO.

Part 2. The rectangle RPL is equal to the square of CZ.

Because DR is to GL, as RP is to PL (*Cor.* 2. 14. 1 *Sup. and* 22. 6 *Eu.*), the rectangle under DR and GL is similar to the rectangle RPL, and is therefore to that rectangle, as the square of DR is to the square of RP (22. 6 *Eu.*), or which is equal (42. 1 *Sup.*), as the square of CO to the square of CZ; whence, the rectangle under DR and GL being equal to the square of CO, by the preceding part, the rectangle RPL is equal to the square of CZ (14. 5 *Eu*).

Part 3. The rectangle XPK is equal to the square of CZ.

Because the rectangle CHK is equal to the rectangle DHG (45. 1. *Sup.*), CH is to HG, as DH is to HK (16. 6 *Eu.*), and therefore, because of the parallels, XP is to PL, as RP is to PK (*Cor.* 2. 10. 6 *and* 11. 5 *Eu.*), and so the rectangle XPK is equal to the rectangle RPL (16. 6 *Eu.*), or, by the preceding part, to the square of CZ.

Cor. 1. In ellipses and hyperboles, a right line joining the contacts of two parallel tangents is a diameter, the right line DG, joining the contacts of the parallel tangents DR and GL, being in the demonstration of the 1st part of this prop. proved to be a diameter.

Cor. 2. If the right line XPK touching an ellipse, or hyperbola, meet two diameters CO and DG, and the rectangle XPK be equal to the square of the semidiameter CZ, conjugate to that which passes through the contact P; the diameters CO and DG are conjugate ones.

For if any other semidiameter, except CG, were conjugate to CO, the rectangle under XP, and a right line greater or less than PK, would be equal to the square of CZ (*part* 3 *of this prop.*), contrary to the supposition.

PROP. XLVIII. PROB.

To draw a tangent to a given conick section, from any point not within the same.

Part 1. If the given point be in the section, having found, in the case of an ellipse or hyperbola, the focuses, and, in the case of a parabola, the focus and principal vertex, by 35. 1 Sup, the tangent may be drawn by Cor. 1. 10. 1 Sup.

Fig. 1.

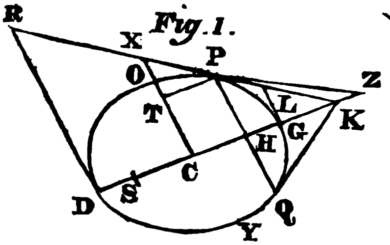


Fig. 2.

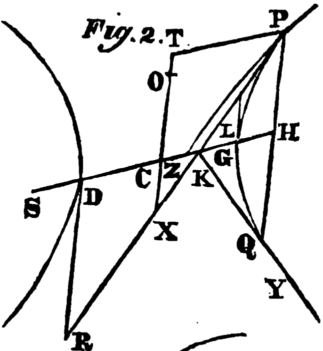


Fig. 3

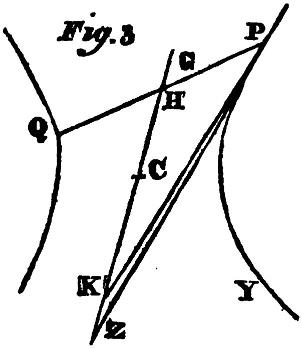
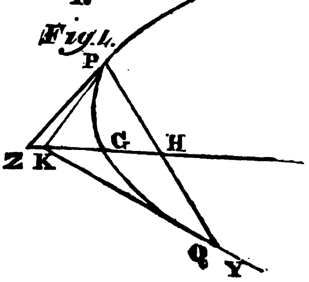


Fig. 4.



Part 2. But if the given point be without the section, not being in the asymptote of a hyperbola, for if it be, the tangent may be drawn by Cor. 6. 37. 1 Sup. Let that point be K, see fig. 1, 2, 3 and 4, and the given conick section PY; through K draw the diameter KG (*Cor.* 35. 1 *Sup.*), meeting the section in G, or, if it be a second diameter, as in fig. 3, let its vertex be G; in the case of an ellipse or hyperbola, find the centre C (33. 1 *Sup.*), and to CK and CG take on the diameter KG, a third proportional CH, to the part contrary to CK, when it is a second diameter of a hyperbola, as in fig. 3, otherwise, as in fig. 1 and 2, to the same part; in the case of a parabola, fig. 4, take GH equal to KG; in all the cases draw through H the right line PHQ ordinately applied to the diameter HK, meeting the section or sections in P and Q; KP and KQ being joined touch the section or sections in P and Q; for if either of them, as KP, were not a tangent to the section, let a tangent PZ, drawn from P by part 1, meet the diameter HK in Z; and in the case of an ellipse or hyperbola, CH, CG and CZ would be continually proportional (44. 1 *Sup.*), and therefore CG would have the same ratio to CK and CZ (*Constr. and* 11. 5 *Eu.*), which is absurd (8. 5 *Eu.*); and in the case of a parabola, fig. 4, GZ would be equal to GH (44. 1 *Sup.*), or its equal by construction GK, which is also absurd (*Ax.* 9. 1 *Eu*).

Scholium. It appears from the construction of this problem, that two tangents may be drawn to a conick section or opposite sections, from any point without it or them, as the case may be, which is not in the asymptote of a hyperbola.

PROP. XLIX. THEOR.

A right line (KH), passing through the concourse (K) of two right lines (PK and QK) touching a conick section or opposite sections, and bisecting the right line (PQ), joining their contacts, is a diameter of the section.

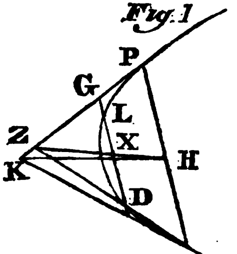


Fig. 1

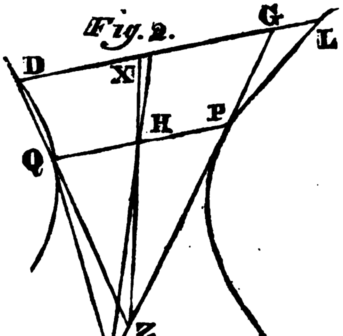


Fig. 2.

For if KH be not a diameter, let a diameter HZ be drawn through H (*Cor.* 35. 1 *Sup.*), meeting the tangent PK in Z, and let QZ be drawn, meeting the section in D, through D let the right line DG be drawn parallel to PQ, meeting, in the case of fig. 1, because D is not the vertex of the diameter HZ, and DG is parallel to the secant PQ, the section again, as in L, and in the case of fig. 2, seeing that DG is parallel to PQ cutting the opposite sections, meeting the opposite section, as in L, and meeting in both cases, ZH in X, and PK in G.

And since PQ, see fig. 1 and 2, is bisected by the diameter HZ, it is ordinately applied to that diameter (32. 1 *Sup.*), therefore DL, parallel to PQ, is ordinately applied to the same diameter, and is therefore bisected by it (31. 1 *Sup.*), and so DX is equal to LX; and because of the equiangular triangles QHZ and DXZ, QH is to HZ, as DX is to XZ (4. 6 *Eu.*), and because of the equiangular triangles ZHP and ZXG, HZ is to HP, as XZ is to XG (*by the same*), therefore, by equality, QH is to HP, as DX is to XG (22. 5 *Eu.*) ; whence, QH being equal to HP, DX is equal to XG (*Cor.* 13. 5 *Eu.*) ; but DX is above proved to be equal to LX, therefore XG and XL are equal (*ax.* 1. 1. *Eu.*), part and whole, which is absurd ; therefore no right line drawn through H, except KH is a diameter of the section, and of course KH is a diameter.

Cor. 1. Hence a diameter of a conick section, passing through the concourse of two tangents, bisects the right line joining their contacts.

For if the diameter divided it unequally, a right line drawn from the concourse to bisect it, being (*by this proposition*), a diameter, there would be two diameters passing through the concourse of the tangents, and therefore two centres, which is absurd.

Cor. 2. If two right lines, touching a conick section or opposite sections, meet each other; their concourse is in the diameter of the section, which bisects the right line joining their contacts.

For the right line joining the contacts is not a diameter, for if it were, the tangents would be parallel (30. 1 *Sup.*), if therefore the diameter, bisecting the right line joining the contacts, did not pass through the concourse of the tangents, the diameter passing through the concourse of the tangents, would not bisect the right line joining the contacts, contrary to the preceding corollary.

PROP. L. THEOR.

If from a vertex of each of two conjugate diameters (CP and CO, see fig. 1 and 2), ordinates (PH and OT), be drawn to a third diameter (DG); the square of the segment (CT), of the third diameter (DG), between the centre and either ordinate in ellipses, or between the centre and that drawn from the vertex of the second diameter in hyperbolas, is equal to the rectangle (DHG), under the segments of that third diameter, between the other ordinate and its vertices. And, in hyperbolas, the square of the segment (CH, see fig. 2), of the same third diameter between the centre and ordinate drawn from the vertex of the transverse diameter (CP), is equal to the sum of the squares of the semidiameter (CG), to which the ordinates are drawn, and the segment (CT), of the same diameter, between the centre and the other ordinate.

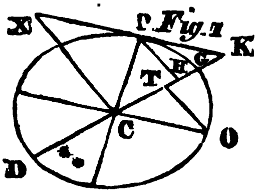


Fig. 1

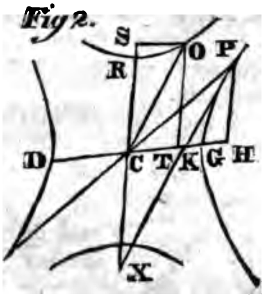


Fig. 2.

Though the vertex P of either of the conjugate diameters draw a tangent XPK (48. 1 *Sup.*), meeting the diameter DG in K, and its conjugate in X; and (*in fig.* 1.), because of the parallels PK and CO, PH and OT, the triangles PHK and OTC are equiangular (*Cor.* 3. 9. 1 *Sup.*), and, because of the parallels CX and PH, the rectangles XPK and CHK are similar (2. 6 *and Def.* 1. 6 *Eu.*); therefore the rectangle XPK is to CHK, as the square of PK is to the square of HK (22. 6 *Eu.*), or, because of the equiangular triangles, as the square of CO is to the square of CT; but the rectangle XPK is equal to the square of CO (47. 1 *Sup.*), therefore the square of CT is equal to the rectangle CHK (14. 5 *Eu.*), or, which is equal (45. 1 *Sup.*), the rectangle DHG under the segments of the diameter DG between the ordinate PH and its vertices. Taking away these equals from the square of CG, the excess of the square of CG above the rectangle DHG, or, which is equal (5. 2 *Eu.*), the square of CH, is equal to the excess of the square of CG above that of CT, or (5. 2 *Eu.*), the rectangle DTG.

And, in the case of fig. 2, the demonstration of the equality of the square of CT to the rectangle DHG, in the preceding paragraph, is applicable to this case and its figure, without variation. Adding to these equals the square of CG, the rectangle DHG with the square of CG, or which is equal (6. 2 *Eu.*), the square of CH, is equal to the squares of CG and CT together.

Cor. Hence in the ellipse, the squares of the segments (CH and CT), of the diameter (DG), to which ordinates are drawn from the vertices of two conjugate diameters, between the centre and ordinates, are together equal to the square of the semidiameter (CG) to which they are so drawn. For since the square of CH is, by this proposition, equal to the rectangle DTG, the square of CH within the square of CT, is equal to the rectangle DTG and the square of CT, or which is equal (5. 2 *Eu.*), the square of CG.

PROP. LI. THEOR.

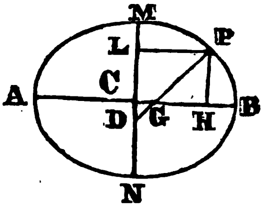
If from a vertex of each of two conjugate diameters (CP and CO, see fig. 2 to the preceding proposition), of a hyperbola, ordinates (PH and OS) be drawn to two other conjugate diameters (CG and CR); the segments (CH and CS) of the diameters, to which the ordinates are drawn, between the centre and the ordinates, are directly, and the ordinates themselves (PH and OS) inversely, as the semidiameters (CG and CR) to which they are drawn.

For the sum of the squares of CG and CT, or which is equal (50. 1 *Sup.*), the square of CH, is to the square of OT or CS, as the square of CG is to the square of CR (40. 1 *Sup.*), therefore CH is to CS, as CG is to CR (22. 6 *Eu*).

And the rectangle DHG, or which is equal (50. 1 *Sup*), the square of CT or OS, is to the square of PH, as the square of CG to the square of CR, (40. 1 *Sup.*), therefore OS is to PH, as CG to CR (22. 6 *Eu*).

PROP. LII. THEOR.

If one extreme (D), of a right line (DP), equal to the principal semiaxis (CB) of an ellipse, be in the second axis (MN), and the segment thereof (DG) between the axes, be equal to the difference of the semiaxes (CB and CM); its other extreme (P) is in the perimeter of the ellipse.



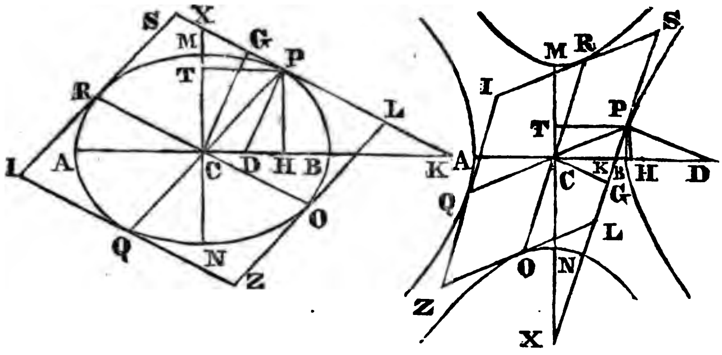
Let fall the perpendiculars PH and PL on CB and CM.

Because of the equiangular triangles PHG and DLP, the square of PH is to the square of GP or CM, as the square of DL, or, (47. 1 *Eu.*), the difference of the squares of DP and LP, or of CB and CH, or, (5. 2 *Eu.*), the rectangle AHB is to the square of DP or CB; therefore the point P is in the perimeter of the ellipse *Cor.* 3. 40. 1 *Sup.*).

Cor. Hence, if two unequal right lines AB and MN, of which AB is the greater, bisect each other at right angles in C, and a right line DG be placed between AB and MN equal to the difference of their halves CB and CM, and on DG produced, GP be taken equal to CM, and the right line DGP be so moved through the four right angles, that the point D may be always in the right line MN, and G in AB; the point P would describe an ellipse, whose transverse axis is AB, and second axis MN. And hence an ellipse is described, by means of an instrument, called *an Elliptick Compass*, as to any one viewing its structure, may hence easily appear.

PROP. LIII. THEOR.

A parallelogram (LSIZ), described about two conjugate diameters (QP and RO), of an ellipse or hyperbola, by drawing through their vertices, four right lines, touching the ellipse or conjugate hyperbolas, is equal to the rectangle under the axes (AB and MN).



Let C be the centre, and SPL produced if necessary, meet AB in K, and MN in X, let PH and PT be drawn, at right angles to the axes, and PD and CG at right angles to SPL.

Then the right angled triangles (see fig. 1), PHD and CGX, being, because of the parallelism of the sides forming the angles DPH and GCX, equiangular (*Cor.* 3. 9. 1 *Sup.*), PH is to PD, as CG is to CX (4. 6 *Eu.*), therefore the rectangle under PD and GC is equal to the rectangle under PH and CX (16. 6 *Eu.*), or to the rectangle TCX, or, which is equal (44. 1 *Sup.* and 17. 6 *Eu.*), the square of CM; and therefore CG, CM and PD are continually proportional (17. 6 *Eu.*); therefore the square CG is to the square of CM as CG is to PD (*Cor.* 2. 20. 6 *Eu.*), or, because of the equiangular triangles CGK and DPK, as GK is to PK, or, which is equal (1. 6 *Eu.*), as the rectangle GKP, or, because of the equiangular triangles CGK and PHK, the rectangle CKH (4 and 16. 6 *Eu.*), or (46. 1 *Sup.*), AKB to the square of PK, or, which is equal (41. 1 *Sup.*), as the square of CB to the square of CO; therefore CG is to CM as CB is to CO (22. 6 *Eu.*), and so the rectangle under CO and CG, or, which is equal (35. 1 *Eu.*), the parallelogram COLP, is equal to the rectangle under CB and CM (16. 6 *Eu.*); whence the parallelogram LSIZ being fourfold the parallelogram COLP (23. 6 *Eu.*), and the rectangle under AB and MN fourfold that under CB and CM (*by the same*), the parallelogram LSIZ is equal to the rectangle under AB and MN.

The reasoning of the preceding paragraph, applies to the case of fig. 2 without variation.

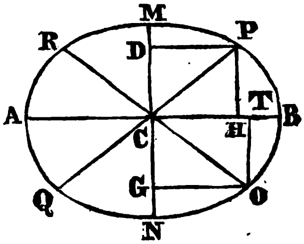
Cor. 1. All parallelograms, described about conjugate diameters, of a given ellipse or hyperbola, by drawing tangents through their vertices, are equal to each other, being each of them (*by this prop.*), equal to the same rectangle.

Cor. 2. All parallelograms, formed by joining the vertices of conjugate diameters of a given ellipse or hyperbola, are equal; being halves of the parallelograms treated of in the preceding corollary.

PROP. LIV. THEOR.

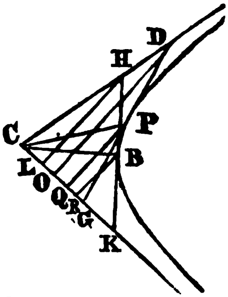
The sum of the squares of any two conjugate diameters of an ellipse, is equal to the sum of the squares of the axes.

And if the angles contained by the asymptotes of a hyperbola be right, any two conjugate diameters are equal. But if the angles contained by the asymtotes be not right, any two conjugate diameters are unequal; and the difference of the squares of any two conjugate diameters, is equal to the difference of the squares of the axes.



Part 1. Let CB and CM be semiaxes of an ellipse, AB and MN being the axes, and CP and CO two other conjugate semidiameters, let PH and OT be perpendicular to CB, and PD and OG to MN.

Because the square of CB is equal to the squares of CH and CT together (*Cor.* 50. 1 *Sup.*), and the square of CM to the squares of CD and CG together (*by the same*), or to those of PH and OT; the squares of CB and CM together are equal to the four squares of CH, CT, PH and OT, to which the squares of CP and CO are also equal (47. 1 *Eu.*), therefore the squares of CP and CO together are equal to those of CB and CM together; but the squares of the diameters QP and RO together are fourfold the squares of CP and CO together (*Cor.* 4. 2 *Eu.*), and the squares of AB and MN together are fourfold the squares of CB and CM together (*by the same*); therefore the squares of QP and RO together are equal to the squares of AB and MN together (*Ax.* 1. 5 *Eu*).



Part 2. Let CD and CK be the asymptotes of a hyperbola, whose centre is C; let CP be any semidiameter drawn to the hyperbola BP, and through its vertex P, let a tangent DPG be drawn (48. 1 *Sup*.), meeting the asymptotes in D and G, PD or PG is equal to the semidiameter conjugate to CP (*Cor*. 2. 37. 1 *Sup*). But if the angle DCG be right, a circle described from the centre P about the diameter DG would pass through C (*Cor*. 31. 3 *Eu*.), and of course the semidiameter CP would be equal to PD or PG, or, by Cor. 2. 37. 1 *Sup*, to the semidiameter conjugate to it, and so the diameter drawn from P would be equal to its conjugate (*Ax*. 6. 1 *Eu*).

In this case, the hyperbola is said to be, *Equilateral* or *Right-angled* (see Def. 20. 1 Sup).

Part 3. Det now the asymptotes CD and CK contain an acute angle, and let CP be any semidiameter; draw the transverse semiaxis CB (35. 1 *Sup*.), through P and B draw the tangents DG and HK, meeting the asymtotes in D and G, H and K; from P, D, B and H, let fall on the asymptote CK, the perpendiculars PQ, DO, BR and HL.

Because the angle DCG is acute (*Hyp*.), it falls without a semicircle described about DG as a diameter, for otherwise the angle GCD would be right or obtuse (31. 3 and 16. 1 *Eu*.), contrary to the supposition, therefore the semidiameter CP of the hyperbola is greater than PD or PG, and therefore than its conjugate semidiameter (*Cor*. 2. 37 and *Def*. 14. 1 *Sup*).

And because DG and HK are bisected in P and B (37. 1 *Sup*.), OG and LK are, because of the parallels, bisected in Q and R (2. 6 *Eu*.); and since the rectangle GCD is equal to the rectangle KCH (*Cor*. 4. 38. 1 *Sup*.), CK is to CG, as CD is to CH (16. 6 *Eu*.), or, because of the parallels, as CO is to CL (2. 6 *Eu*.); therefore the rectangle GCO is equal to the rectangle KCL (16. 6 *Eu*.); but the rectangle GCO is equal to the difference of the squares of CQ and QG (6. 2 *Eu*.) or (*Cor*. 1 and *Schol*. 6. 2 *Eu*.), to the difference of the squares of CP and PG; and for the same reason the rectangle KCL is equal to the difference of the squares of CR and RK, or of CB and BK; therefore the difference of the squares of CP and PG, is equal to the difference of the squares of CB and BK; but PG and BK are equal to the semidiameters which are conjugate to the semidiameters CP and CB (*Cor*. 2. 37. and *Def*. 14. 1 *Sup*.), therefore the differences of the squares of the semidiameters CP and CB, and their conjugates, are equal, and therefore the difference of the squares of the diameter drawn from P, and its conjugate, is equal to the difference of the squares of the axes.

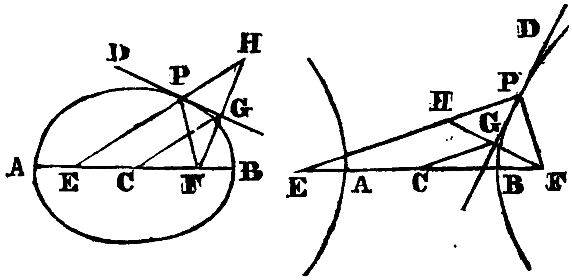
If the angle GCD were obtuse, the angle formed by the asymptotes, towards a hyperbola conjugate to the hyperbola BP, would be acute (13. 1 *Eu.*), and so the proof of the case here demonstrated, applies also to that case.

Scholium. From the demonstration of the 3d part of this prop. it follows, that, in the case of the asymptotes of hyperbolas, cutting each other at oblique angles, any diameter, terminated by the opposite hyperbolas, which are included within the acute angles, is greater than its conjugate.

Cor. In ellipses, the sum, and in hyperbolas, the difference of the squares, of any two conjugate diameters, is equal to the sum or difference, as the case may be, of the squares of any other conjugate diameters, being each, by this prop. equal to the sum or difference of the squares of the axes.

PROP. LV. THEOR.

A right line, drawn from the centre of an ellipse or hyperbola, to a tangent to the section, parallel to a right line, joining a focus and the contact, is equal to the transverse semiaxis.



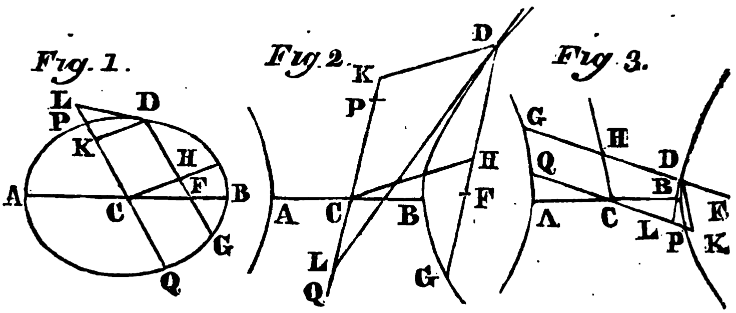
Let CG, see both fig. be a right line, drawn from the centre C of an ellipse or hyperbola, to a tangent DG, parallel to a right line EP, joining a focus E to the contact P, AB being the transverse axis; CG is equal to AC or CB.

Having found the other focus F (35. 1 *Sup*), draw to it PF, join FG, which produce to meet EP in H. Because then EH and CG are parallel (*Hyp.*), and EC and CF equal [*Def.* 3 and

5. 1 *Sup.*], HG and GF are also equal [2. 6 *Eu.*] ; and, because the tangent PG bisects the angle FPH [11. 1 *Sup.*], FG is to GH, as FP is to PH [3. 6 *Eu.*], therefore FP and PH are equal [*Cor.* 13. 5 *Eu.*], and so EH is equal to AB [1. 1 *Sup.*] ; but, because the triangles CFG and EFH are equiangular, and CF is the half of EF, CG is the half of EH or AB [4. 6 and 16. 5 *Eu.*], and therefore equal to AC or CB.

PROP. LVI. THEOR.

The segment of a right line, drawn through the focus of an ellipse or hyperbola, intercepted by the section or hyperbolas, is a third proportional, to the transverse axis, and the diameter, parallel to the right line so drawn.



Let DG be the segment of a right line, drawn through the focus F of an ellipse or hyperbola, intercepted by the section or opposite hyperbolas, see fig. 1, 2 and 3, and PQ, a diameter parallel thereto, AB being the transverse axis ; DG is a third proportional to AB and PQ.

Let C be the centre, and let a diameter be drawn bisecting DG in H [*Cor.* 35. 1 *Sup.*], this diameter is conjugate to PQ [32. 1 and *Def.* 12 and 14. 1. *Sup.*], from D draw the ordinate DK to the diameter PQ [36. 1 *Sup.*], this is parallel to CH [*Def.* 12 and 14. 1 *Sup.*], draw the tangent DL [48. 1 *Sup.*], meeting PQ in L ; and because CL, drawn from the centre to the tangent, parallel to DF, joining the contact and a focus, is equal to CB [55 1 *Sup.*], and DG is bisected in H [*constr.*], and therefore DH or CK is equal to the half of DG, and CL, CP and CK are continually proportional [44. 1 *Sup.*], therefore AB, PQ and DG, which are double to CL, CP and CK, are continually proportional, and so DG is a third proportional to AB and PQ.

Cor. Hence, the rectangle under the transverse axis, and a right line terminated both ways by an ellipse, hyperbola or opposite hyperbolas, and, being produced if necessary, passing through the focus, is equal to the square of the diameter parallel thereto [*by this prop. and* 17. 6 *Eu.*] ; and therefore, the transverse axis being constant, right lines so terminated are to each other, as the squares of the diameters, to which they are parallel [1. 6 and 11. 5 *Eu.*], or in a duplicate ratio of those diameters [20. 6 *Eu*].

PROP. LVII. THEOR.

Right lines terminated both ways by a conick section or opposite sections, and, being produced if necessary, passing through the focus, are to each other, in the case of an ellipse or hyperbola, in a ratio compounded of the ratios of the diameters, to which they are ordinately applied, and of their parameters ; and, in the case of a parabola, in the ratio of the parameters of the diameters, to which they are so applied.

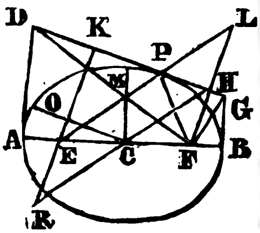
In the case of an ellipse or hyperbola, these right lines are to each other, as the squares of the diameters to which they are parallel [*Cor.* 56 *of this*], and therefore, the rectangle under any diameter and its parameter, being equal the square of its conjugate [*Def.* 15. 1 *Sup.* and 17. 6 *Eu.*], in a ratio compounded of the ratios of the diameters, whose ordinates are parallel to these right lines, and their parameters [23. 6 *Eu*].

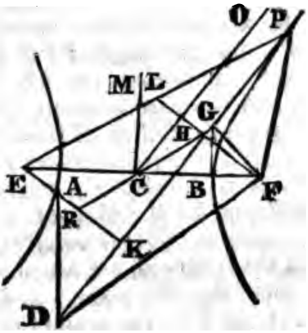
In the case of a parabola, these right lines are equal to the parameters of the diameters whose ordinates are parallel to them [39. 1 *Sup.*], and are therefore to each other as these parameters [7. 5 *Eu*].

Cor. Hence these right lines are to each other, as the rectangles under their segments, between the focus and the section or sections, since these rectangles are to each other, in the ratio specified in this proposition [*Schol.* 42. 1. *Sup*].

PROP. LVIII. THEOR.

If tangents, drawn through the extremes of the transverse axis of an ellipse or hyperbola, meet a third tangent; a circle, described about the segment of the third tangent, intercepted by the other tangents, as a diameter, passes through the focuses of the section.





Let two right lines AD and BG, touching an ellipse or hyperbola in the extremes A and B of the transverse axis, meet a third tangent DG, a circle described about the segment DG of the third tangent, intercepted between AD and BG, as a diameter, passes through the focuses E and F.

Let P be the point, in which DG touches the section, and CM, the second semiaxis; the rectangle under AD and BG is equal to the square of CM [47. 1 *Sup.*], or, which is equal [2. 1 *Sup.*], the rectangle AFB, therefore BG is to BF, as AF is to AD [16. 6 *Eu.*], and these right lines are about equal angles FBG and FAD, both these angles being right ones, therefore, the triangles FBG and FAD are equiangular [6. 6 *Eu.*], therefore the angles FCB and DFA are equal; but, because the angle FBG is right, the angles FGB and BFG are together equal to a right angle [32. 1 *Eu.*], therefore the angles DFA and GFB are together equal to a right angle, and therefore the angle DFG is right, and of course the point F is at the circumference of a circle, described about DG as a diameter [*Cor* 31. 3 *Eu*]. In like manner it may be proved, that E is at the circumference of the same circle.

PROP. LIX. THEOR.

Right lines, drawn from any point of an ellipse or hyperbola, to the focuses contain a rectangle, equal to the square of the semidiameter, which is parallel to a tangent, drawn through that point.

Let PE and PF, see figures to preceding prop, be right lines, drawn from a point P of an ellipse or hyperbola, to the focuses, and CO be a semidiameter, parallel to a tangent DG drawn through P; the rectangle EPF is equal to the square of CO.

Draw the transverse axis AB [35. 1 *Sup.*], and through its vertices A and B, draw tangents to the section AD and BG [48. 1 *Sup.*], meeting DG in D and G: from either focus as F, let fall on DG, the perpendicular FH, which produce to meet EP in L; and, because the triangles FHP and LHP have the angles FPH and LPH equal (11. 1 *Sup.*], the angles at H right, and PH common, FH is equal to HL, and PF to DL [26. 1 *Eu.*]; and the centre of a circle described about DG as a diameter is in DG [*Def.* 13. 1 *Eu.*], and right lines drawn from that centre to F and L are equal [4. 1 *Eu.*]; since therefore that circle would pass through the focus E and F [58. 1 *Sup.*], it would also pass through the point L; therefore, because of the circle, the rectangle EPL or EPF is equal to the rectangle DPG [35 and 36. 3 *Eu.*], or, which is equal [47. 1 *Sup.*], to the square of CO.

PROP. LX. PROB.

If from the focuses of an ellipse or hyperbola, perpendiculars be let fall on any tangent; a circle described about the transverse axis, as a diameter, passes through the points, in which the perpendiculars meet the tangent.

Let EK and FH, see figures to prop. 58, be perpendiculars let fall from the focuses E and F, of an ellipse or hyperbola, on a tangent DG, and AB be the transverse axis; a circle described about AB, as a diameter, passes through the points K and H, in which the perpendiculars meet DG.

Let C the centre, and P the point, in which DG touches the section; join EP and PF, let the perpendicular FH produced meet EP in L, and join CH. Because the angles FPH and LPH are equal [11. 1 *Sup.*], PH common to the triangles FPH and

LPH, and the angles at H right, FH is equal to HL, and PF to PL [26. 1 *Eu.*], therefore EL is equal to the transverse axis AB [1. 1 *Sup.*]; and since FH and HL are equal, and FC equal CE [*Def.* 3 and 5. 1 *Sup.*], EL and CH are parallel [2. 6 *Eu.*]; and since the triangles CFH and EFL are equiangular, and CF is the half of EF, CH is the half of EL or AB [4. 6 and 16. 5 *Eu.*]; therefore the point H is in the circumference of a circle, described about AB as a diameter. In like manner it may be proved, that the point K is in the same circumference.

PROP. LXI. THEOR.

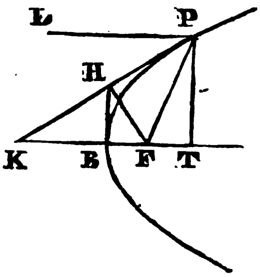
The rectangle under perpendiculars, let fall from the focuses of an ellipse or hyperbola, on any tangent, is equal to the square of the second semiaxis.

Let EK and FH, see figures to prop. 58, be perpendiculars, let fall from the focuses E and F of an ellipse or hyperbola, on a tangent DG, and CM the second semiaxis, C being the centre, the rectangle under EK and FH is equal to the square of CM.

Let AB be the transverse axis, join CH, which produce to meet KE in R; because of the parallels FH and ER, the triangles CFH and CER are equiangular, and therefore, because of the equals EC and CF, the right lines CH and CR, and FH and ER are equal [26. 1 *Eu.*]; but a circle described about AB as a diameter, passes through the points H and K [60. 1 *Sup.*], and, because of the equals CH and CR, it passes also through R; therefore, because of the circle described as above, the rectangle KER, or, ER and FH being equal, the rectangle under EK and FH, is equal to the rectangle AEB, or, which is equal [2. 1 *Sup.*], to the square of CM.

PROP. LXII. THEOR.

A perpendicular, let fall from the focus of a parabola, on a tangent, is a mean proportional, between the distances of the focus, from the point of contact, and from the principal vertex.



Let F be the focus, and B the principal vertex of a parabola, PK a tangent meeting the axis in K, P being the point of contact, and FH a perpendicular let fall from F on PK; FH is a mean proportional between FP and FB.

Draw the ordinate PT to the axis (36. 1 *Sup.*), through P draw PL parallel to KT, and join HB; the angle FPK is equal to LPK (11. 1 *Sup.*), or its alternate PKF, therefore FP is equal to FK (6. 1 *Eu.*); whence, FH being common to the triangles PFH, and KFH, and the angles at H right, PH is equal to HK (*Cor.* 7. 6 *Eu.*), and KT is bisected in B (44. 1 *Sup.*), therefore BH is parallel to TP (2. 6 *Eu.*), and of course perpendicular to the axis, and therefore, because of the right angle FHK, FH is a mean proportional between FK or FP, and FB (*Cor.* 2. 8. 6 *Eu*).

PROP. LXIII. THEOR.

The square of a perpendicular, let fall from a focus of a conick section, on a tangent to the section, is, in the case of an ellipse or hyperbola, to the rectangle, under the distances of the same focus from the contact, and the nearer principal vertex, as the distance of the same focus from the other principal vertex, to the distance of the other focus from the contact; and, in the case of a parabola, in the ratio of equality.

Fig. 1.

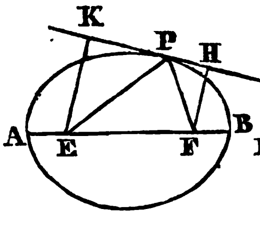


Fig. 2.

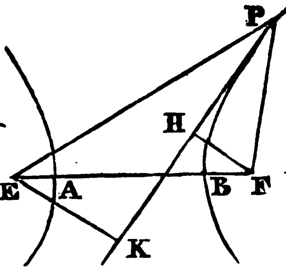
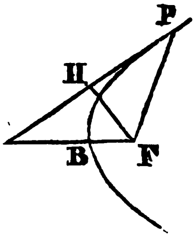


Fig. 3.



Let FH, see fig. 1, 2 and 3, be a perpendicular, let fall from the focus F of a conick section, on a tangent PH, P being the point of contact, and B the principal vertex nearer to F; and in the ellipse and hyperbola, fig. 1 and 2, let A be the other principal vertex, and E the other focus. The square of FH is to the rectangle AFB, in the case of the ellipse and hyperbola, as AF is to EP, and, in that of the parabola, in a ratio of equality.

Let EK in fig. 1 and 2, be a perpendicular, let fall from E on the tangent PH. The right angled triangles FHP and EKP, having the angles at P equal (11. 1 *Sup.* and 15. 1 *Eu.*), are similar, therefore the rectangles under FH and EK, and under FP and EP are similar; therefore the square of FH is to the square of FP, as the rectangle under FH and EK, or (61. 1 *Sup.*), the square of the second semiaxis, or (2. 1 *Sup.*), the rectangle AFB, is to the rectangle FPE (20. 6 and *Cor.* 3. 22. 5 *Eu.*), and, alternating, the square of FH is to the rectangle AFB, as the square of FP is to the rectangle FPE (16. 5 *Eu.*), or, FP being a common side, as FP is to PE (1. 6 *Eu.*), or, as the rectangle PFB is to the rectangle under EP and FB (*by the same*), and, alternating, the square of FH is to the rectangle PFB, as the rectangle AFB is to the rectangle under EP and FB, or, FB being a common side in the two last terms, as AF is to EP (1. 6 *Eu*).

And since, in the case of a parabola, fig. 3, FH is a mean proportional between FP and FB (62. 1. *Sup.*), the square of FH is equal to the rectangle PFB (17. 6 *Eu.*), and is therefore to that rectangle, in a ratio of equality.

Cor. 1. Since, in the case of a parabola, fig. 3, the square of FH is equal to the rectangle BFP, and BF is a constant quantity, the square of FH is as FP; or, the squares of perpendiculars, let fall from the focus of a parabola on tangents, are to each other, as the distances of the points of contact from the focus.

Cor. 2. And since, in the ellipse and hyperbola, fig. 1 and 2, the square of FH is to rectangle PFB, as AF is to EP, and AF and FB are constant quantities; therefore the square of FH is in a ratio, compounded of the direct ratio of FP, and the inverse one of EP, (or as $\frac{FP}{EP}$); or, the squares of perpendiculars, let fall from a focus of an ellipse or hyperbola, on tangents, are to each other, in a ratio, compounded of the direct ratio of the distance of the same focus from the contact, and the inverse one of the distance of the other focus from the same contact. But, in the ellipse, because the sum of EP and PF is a constant quantity, (1. 1 *Sup.*), while FP is increased, EP is diminished, and the contrary; therefore, in the ellipse, the square of FH is more varied, than in the ratio of FP: but, in the hyperbola, because the difference of EP and PF is a constant quantity (1. 1 *Sup*), EP and PF are increased or diminished together; therefore, in the hyperbola, the square of FH is less varied, than in the ratio of FP; and therefore the perpendicular let fall from a focus on the tangent varies, in the ellipse, more, and in the hyperbola, less, than in the subduplicate ratio of the distance of the focus from the contact.

See Cor. 6. 16. 1 *Newt. Princip.*

PROP. LXIV. THEOR.

If two right lines, touching a conick section or opposite sections, meet each other, and a right line, drawn from a point in one of the tangents, parallel to the other, meet the right line joining the contacts, and cut the section or sections in two points; the square of the segment of the secant, between the tangent, and the right line joining the contacts, is equal to the rectangle under the segments of the same, between the tangent, and the section or sections.

Fig. 1.

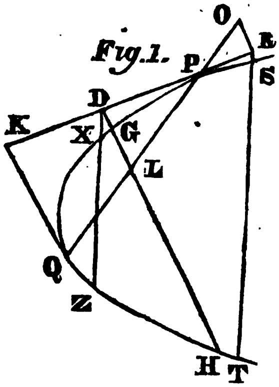


Fig. 2.

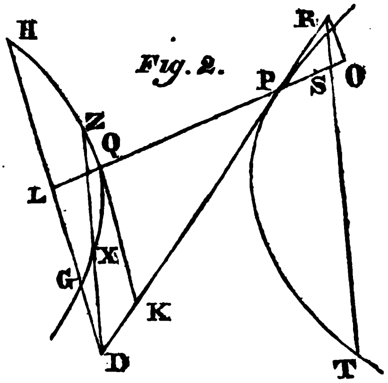
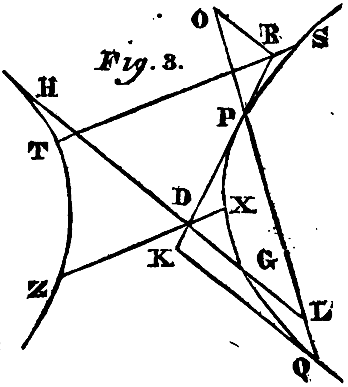


Fig. 3.



Let PK and QK, see allthe fig. of this prop, touching a conick section or opposite sections in P and Q, meet each other in K, and from any point D, in one of the tangents PK, let a right line DH be drawn, parallel to the other tangent QK, meeting the right line PQ joining the contacts in L, and the section or sections in G and H; the square of DL is equal to the rectangle GDH.

For because the tangent PK meets the parallels DH and KQ, the rectangle GDH is to the square of KQ, as the square of PD is to the square of PK (*Cor.* 3. 14. 1 *Sup.*), or, because of the equiangular triangles PDL and PKQ, as the square of DL to the square of KQ (4 and 22. 6 *Eu.*); whence, the square of DL and the rectangle GDH, having the same ratio to the square of KQ, are equal (9. 5 *Eu*).

PROP. LXV. THEOR.

If two right lines, touching a conick section or opposite sections, meet each other, and from any point in one of the tangents, two right lines be drawn, one parallel to the other tangent, to the right lines joining the contacts, and the other in any manner, cutting in two points the section or sections; the square of the right line drawn to that joining the contacts, is to the rectangle under the segments of the secant, between the tangent and the section or sections, in the given ratio of the squares of the segments of tangents, or rectangles under the segments of secants, parallel to these right lines, between their concourse, and the section or sections.

Let PK and QK, see all the fig. of the prec. prop, touching a conick section or opposite sections in P and Q, meet each other in K, and from any point R, in one of the tangents KPR, let two right lines RO and RST be drawn, one RO parallel to the other tangent QK, to the right QPO joining the contacts, and the other RST in any manner, cutting the section or sections in S and T; the square of RO and the rectangle SRT are to each other, in the given ratio of the squares of the segments of tangents, or rectangles under the segments of secants, parallel to RO and RT, between their concourse, and the section or sections.

From any point D in the tangent PK, draw right lines DH and DZ parallel to RO and RT, cutting the section or sections in G and H, X and Z, and let DH parallel to RO, meet the right line joining the contacts in L; the rectangle SRT is to the rectangle XDZ, as the square of PR is to the square of PD (*Cor.* 1. 14. 1 *Sup.*), or, because of the equiangular triangles PRO and PDL, as the square of RO is to the square of DL (4 and 22. 6 and 16. 5 *Eu.*), or its equal (64. 1 *Sup.*), the rectangle GDH; therefore, by alternating and inverting, the square of RO is to the rectangle SRT, as the rectangle GDH is to the rectangle XDZ (16. 5 and *Theor.* 3. 15. 5 *Eu.*), or in the given ratio of the squares of the segments of tangents, or rectangles under the segments of the secants, parallel to RO and RT, between their concourse and the section or sections (42. 1 *Sup*).

PROP. LXVI. THEOR.

If through the concourse (K), of two right lines (PK and QK), touching a conick section or opposite sections, there be drawn a right line, meeting the section or sections in two points, and the right line (PQ) joining the contacts; the right line so drawn is cut harmonically in the concourse of the tangents, the points in which it meets the section, and that, in which it meets the right line joining the contacts.

Fig. 1.

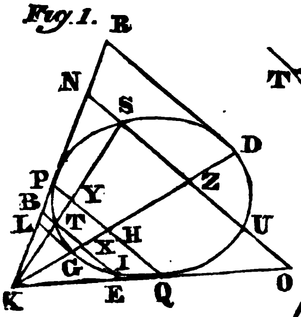
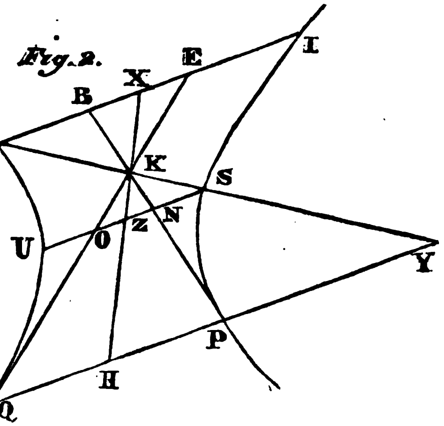


Fig. 2.



Case 1. Let the right line drawn through K, see both fig. as KTS, not be a diameter, and through T and S let BTE and NSO be drawn parallel to PQ, meeting the tangents in B and E, N and O, and the section or sections in T and I, S and U, through K let the diameter KH be drawn (*Cor.* 35. 1 *Sup.*), meeting the right lines BE, PQ and NO in X, H and Z, and, because it bisects the right line PQ in H (*Cor.* 1. 49 1. *Sup.*), it bisects BE and NO in X and Z (4. 6 and 22. 5 *Eu.*), and because TI and SU, terminated by the section or sections, are parallel to PQ, they are bisected in X and Z (*Cor.* 1. 32. 1 *Sup.*); therefore the segments EI and TB and the segments OU and SN are equal, and the rectangles BTE and NSO severally equal to the rectangles TBI and SNU.

And because of the parallels BE and NO, the right line NS is to BT, as SO to TE, therefore the rectangles NSO and BTE are similar; and therefore these rectangles, or the rectangles SNU and TBI are to each other, as the squares of NS and BT

(22. 6 *Eu.*), or, because of the equiangular triangles KNS and KBT, as the squares of KN and KB; but the same rectangles SNU and TBI are to each other, as the squares of PN and PB (*Cor.* 1. 14. 1 *Sup.*); therefore the squares of KN and KB are to each other, as the squares of PN and PB (11. 5 *Eu.*), and therefore KN is to KB, as PN is to PB (22. 6 *Eu.*), and of course, because of the parallels, KS is to KT, as YS is to TY; therefore the right line KS is cut harmonically in the points K, T, Y and S (*Def.* 24. 1 *Sup*).

Case 2. When the right lines PK and QK (see fig. 1) touch the same section, and the right line KGD drawn through K, is a diameter, let it meet PQ in H, and the section or opposite sections in G and D; the right lines GL and DR drawn through these points parallel to PQ are tangents (*Cor.* 1. 49. 1, 32. 1 and *Def.* 12. 1 *Sup.*); let them meet the tangent KR in L and R; and because of the parallels, KR is to KL, as DR to GL, or which is equal (*Cor.* 2. 14. 1 *Sup.* and 22. 6 *Eu.*), as PR to PL; therefore, because of the parallels, KD is KG, as HD is to GH, and of course, the diameter drawn through K, is cut harmonically in the points K, G, H and D (*Def.* 24. 1 *Sup*).

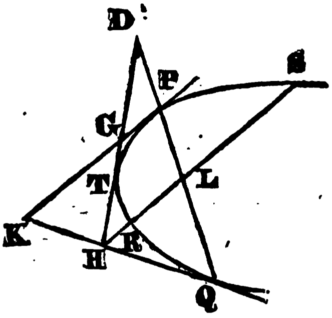
Cor. From the demonstration of this proposition, it follows, that a tangent (KR), which meets two parallel tangents (GL and DR) and the right line (DG) joining their contacts, is cut harmonically in the contact (P) and the points (R, L and K), in which it meets the tangents, and the right line KGD joining their contacts.

Schol. The reasoning in case 2 and the cor. applies to fig. 2, 48, 1 Sup.

PROP. LXVII. THEOR.

Each of three tangents to a conick section or opposite sections, which meets the other two, and the right line joining their contacts, is cut harmonically, in the point of contact, and the points, in which it meets the other tangents, and the right line joining their contacts.

If two of the tangents be parallel, and the third meet the right line joining their contacts, the proposition is manifest from the preceding corollary.



But if the three right lines PK, QK and GH, touching the conick section or opposite sections in P, Q and T, meet each other in K, G and H, and PQ joining the contacts of two of them PK and QK, meet the third GH, produced if necessary, as in D; DH is to DG, as TH to TG.

Through H, the intersection of the tangents TH and QH, let a right line HRS be drawn, parallel to the other tangent KP, meeting the section or each section in R and S, and PQ in L; the square of HL is equal to the rectangle RHS (64. 1 *Sup.*); but, because of the parallels GP and HL, the square of DH is to the square of DG, as the square of HL, or which is equal (64. 1 *Sup.*), the rectangle RHS is to the square of GP (4 and 22. 6 and 16. 5 *Eu.*), or, which is equal (*Cor.* 3. 14. 1 *Sup.*), as the square of TH to the square of GT; therefore DH is to DG, as TH is to TG (22. 6 *Eu*).

PROP. LXVIII. THEOR.

If from any point of a conick section or opposite sections, right lines parallel to two adjacent sides of a quadrangle inscribed in the section or sections, meet the opposite sides of the quadrangle, produced if necessary, the rectangles under the segments of these right lines, between the point in the section, and those opposite sides, are to each other, in the case of an ellipse or hyperbola, as the squares of the semidiameters to which they are parallel, and, in the case of a parabola, as the parameters of the diameters, whose ordinates are parallel to them.

Fig. 1.

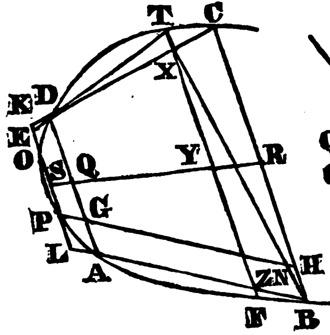
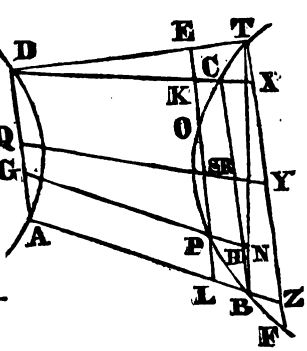


Fig. 2.



Part 1. Let the inscribed quadrangle be a trapezium ABCD, see fig. 1 and 2, having two of the opposite sides AD and BC parallel; and from any point P in the section, let two right lines PK and PH be drawn, parallel to the adjacent sides AD and AB of the trapezium, meeting its opposite sides in the points K and L, G and H; the rectangle KPL and GPH are to each other, in the case of an ellipse or hyperbola, as the squares of the semidiameters to which PK und PH are parallel, and, in the case of a parabola, as the parameters of the diameters, whose ordinates are parallel to these right lines.

For let KPL meet the section again in O, and let QR be drawn, bisecting the parallels AD and BC, and meeting KL in S, it is a diameter of the section (*Cor.* 2. 32. 1 *Sup.*), and bisects in S, as well the right line PO terminated by the section (*Cor.* 1. 32. 1 *Sup.*), as KL, terminated by the right lines KC and LB, and parallel to the bisected right lines; therefore KP and OL are equal, and the rectangle KPL is equal to the rectangle

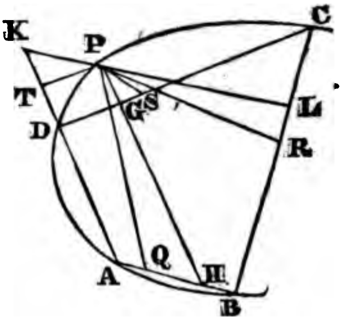
PLO; and, because of the parallelograms, the rectangle GPH is equal to the rectangle ALB: therefore the rectangles KPL and GPH are to each other, as the rectangles PLO and ALB (*Cor.* 1. 7. 5 *Eu.*), or, which is equal (42. 1 *Sup.*), in the case of an ellipse or hyperbola, as the squares of the semidiameters which are parallel to PK and PH, and, in the case of a parabola, as the parameters of the diameters, whose ordinates are parallel to these right lines.

Part 2. Let now the inscribed quadrangle be a trapezium ABTD, none of whose sides are parallel; and from a point P in the section, let two right lines PE and PN be drawn parallel to AD and AB, meeting the opposite sides of the trapezium in the points E and L, G and N; the rectangles EPL and GPN are to each other, as the squares or parameters mentioned.

Through B, draw BC parallel to AD, meeting the section again in C, and PN in H; let CD being joined meet PE in K, and through T, draw TF parallel to AD, meeting the section again in F, and DC and AB in X and Z; draw QR bisecting the parallels AD and BC, and meeting TF in Y, it is a diameter of the section (*Cor.* 2. 32. 1 *Sup.*), and bisects in Y as well the right line TF terminated by the section (*Cor.* 1. 32. 1 *Sup.*), as XZ terminated by the right lines AB and DC, and parallel to the bisected right lines, therefore TX and ZF are equal: and, because of the similar triangles DKE and DXT, the right line KE is to XT or ZF, as DK is to DX, or, because of the parallels KL, DA and TF, as LA or PG is to AZ; and, because of the similar triangles BHN and TZB, the right line BH or PL is to TZ, as NH is to ZB; therefore the rectangle under KE and PL is to the rectangle TZF, as the rectangle under PG and NH is to the rectangle AZB, (23. 6, 22. 5 and *Def.* 13. 5 *Eu.*); therefore, by alternating, the rectangle under KE and PL is to the rectangle under PG and NH, as the rectangle TZF to the rectangle AZB (16. 5 *Eu.*), or, which is equal (14. 1 *Sup and Case* 1 *of this prop.*), as the rectangle KPL is to the rectangle GPH; therefore, by taking the differences of the homologous terms, in the case of fig. 1, and their sums, in that of fig. 2, the rectangle EPL is to the rectangle GPN, as the rectangle KPL is to the rectangle GPH (19 and 12. 5 *Eu.*), or, which is equal, by case 1 of this prop, as the squares or parameters mentioned.

PROP. LXIX. THEOR.

If from any point, in a conick section or opposite sections, four right lines be drawn, to the four sides of a quadrangle inscribed therein, in given angles; the rectangle under those drawn to any two opposite sides, is to the rectangle under those drawn to the other opposite sides, in a given ratio.



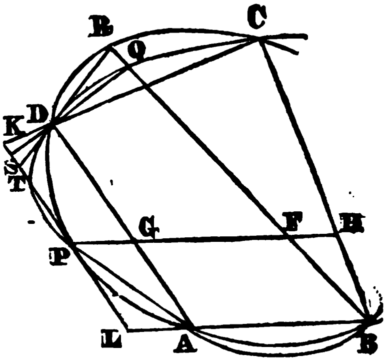
Let ABCD be a quadrangle, inscribed in a conick section, and from any point P of the section, to its four sides AB, BC, CD and DA, let four right lines PQ, PR, PS and PT be drawn, making with these sides given angles, the rectangles QPS and TPR, under the right lines drawn to the opposite sides, are to each other, in a given ratio.

From the point P draw right lines PH and PK, parallel to two adjacent sides AD and AB of the quadrangle, and let them meet its opposite sides in the points G and H, K and L. And because, in the triangle PQH, the angle PQH is given (*Hyp.*), as also the angle PHQ, being the complement of the given angle DAB to two right angles, all the angles of the triangle PQH are given (32. 1 *Eu.*), therefore the ratio of PQ to PH is given (4. 6 *Eu.*); in like manner, in the triangle PGS, the ratio of PS to PG is given, and therefore also the ratio of the rectangle QPS to the rectangle GPH (23 and 12. 6 and 22. 5 *Eu*).

In like manner it may be shewn, that the ratio of the rectangle KPL to the rectangle TPR is given. But the ratio of the rectangle GPH to KPL is given (68, 33 and 35. 1 *Sup*). Since then, the ratio of the rectangle QPS to GPH is given, as also those of GPH to KPL and of KPL to TPR; the ratio compounded of these given ratios, being that of the rectangle QPS to the rectangle TPR, is also given (12. 6 and 22. 5*Eu*).

PROP. LXX. THEOR.

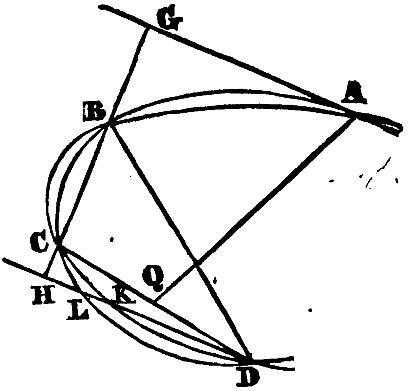
A conick section, does not meet a conick section, or opposite sections, in more points than four.



For, if possible, let two conick sections CQD and CRD meet each other in more points than four, as in the points A, B, C, D and P; let the quadrangle ABCD be formed by connecting four of these points, and from the fifth point P let PH and PK be drawn, parallel to two adjacent sides AB and AD, meeting the opposite sides of the quadrangle in G and H, K and L; let BR be drawn cutting in any manner the sections CQD and CRD in Q and R, and meeting PH in F; and let QD and RD being joined, meet KL in S and T. Then since the rectangle KPL is to GPH, as the squares of the diameters parallel to KL and PH, or the parameters of the diameters whose ordinates are parallel thereto, as the case may be (68. 1 *Sup.*), these squares or parameters are in the same ratio to each other, in both the sections CQD and CRD (11. 5 *Eu*); but as well the rectangle SPL as the rectangle TPL is to the rectangle GPF, as the same squares or parameters (68. 1 *Sup.*), therefore the rectangle SPL is to GPF, as TPL to the same GPF (11. 5 *Eu.*), and so the rectangles SPL and TPL are equal (9. 5 *Eu.*), which is absurd (1. 2 *Eu.*), therefore the sections CQD and CRD cannot meet each other in more than four points.

PROP. LXXI. THEOR.

If two conick sections touch each other, they do not meet each other in three other points, besides the point of contact.



Let two sections touch each other in the point A, they do not meet each other in three other points.

For, if possible, let them meet each other in three other points B, C and D; through A let the common tangent to both sections GA be drawn (48. 1 and *Def.* 21. 1 *Sup.*), and let the points B, C and D be joined by three right lines; and first, let none of them be parallel to GA; produce one of them BC to meet GA in G, and from D draw a right line HD parallel to GA, meeting GBC in H, and the sections in K and L; then each of the rectangles DHK and DHL is to the square of GA, as the rectangle CHB to the rectangle CGB (*Cor.* 6. 14. 1 *Sup.*); therefore the rectangles DHK and DHL are equal (11 and 9. 5 *Eu.*), and so the points L and K coincide, and the sections meet each other in five points, contrary to the preceding proposition. Or if DH touch one section in D, and meet the other again in L, it may be shewn in the same manner, that the square of DH is equal to the rectangle DHL, which is absurd (2. 2 *Eu*). Or if DH touch both sections in D, because it is parallel to the tangent GA, DA being joined would be a diameter of each section (*Cor.* 1. 47. 1 *Sup.*), and therefore, if from the common point B, there be inscribed in one of the sections a right line parallel to GA, and of course ordinately applied to the common diameter (*Def.* 12. 1 *Sup.*), and its other extreme would be also in the other section, and the ratio of the square of an ordinate drawn to the same diameter, from any other point C in one of the sections, to the square of an ordinate drawn from B to the same, is equal to the ratios of the rectangles under the abscissas (*Cor.* 2 40. 1 *Sup.*), and therefore C is also in the other section (*Cor.* 3. 40. 1 *Sup.*), and so the sections would meet each other in five points, which is absurd (70. 1 *Sup*).

But if CD, joining two points common to each section, be parallel to the common tangent GA, it is ordinately applied to the diameter passing through the contact A (*Def.* 12. 1 *Sup.*), which diameter therefore bisects CD (31 1 *Sup.*), and therefore a right line AQ, drawn from the contact A, bisecting CD in

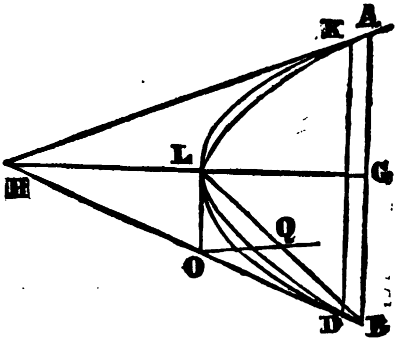
Q, is a diameter of each section, for if a diameter of either drawn through A, (which would bisect CD by 31. 1 *Sup.*), were different from AQ, the right line CD would be bisected in two points, which is absurd; if therefore, from the point B, common to both sections, there be inscribed in one of the sections, a right line parallel to GA or CD, and of course ordinately applied to the common diameter AB (*Def.* 12. 1 *Sup.*), its other extreme would also be in the other section, and so the sections would meet in five points, which is absurd (70. 1 *Sup*). Therefore the sections do not meet in three points, besides the contact.

Cor. A right line, passing through the contact of a tangent to a conick section, and bisecting a right line parallel to the tangent, and terminated by the section or opposite sections, is a diameter; it being demonstrated in this proposition, that AQ, drawn from the contact A, of a tangent to a conick section, and bisecting CD, is a diameter.

PROP. LXXII. THEOR.

If two conick sections touch each other in two points, they do not meet in any other point.

Let two conick sections touch each other in A and B; and first, let the common tangents, (48. 1 and *Def.* 21. 1 *Sup.*) drawn through A and B meet each other in H, through the concourse of the tangents, draw HG bisecting the joined right line AB, this is a diameter of each section (49. 1 *Sup.*), and AB ordinately applied to it (32. 1 *Sup.*), if therefore these sections have any other common point as D, let DK be drawn ordinately applied to the common diameter, HG, it meets each section in the other extreme K, and so these sections meet in three other points B, D and K, besides their contact A, contrary to the preceding proposition, or if a point L in the diameter HG were common to each section, a right line LO drawn through L parallel to AB would touch each section in the point



L (*Def.* 12. 1 *Sup.*), let this tangent meet the tangent HB in O; and, BL being joined, the right line OQ drawn through O and bisecting BL is a common diameter of each section (49. 1 *Sup.*), and if therefore from the remaining common point A, a right line be drawn ordinately applied to this diameter its other extreme is in each section, contrary to the preceding prop, as before; there is therefore no point common to the sections, besides the contacts A and B.

But if the common tangents through A and B do not meet, being parallel, AB being joined is a diameter of each section (*Cor.* 1. 47. 1 *Sup.*); if therefore, they had another common point besides the contacts, a right line being drawn from this common point, ordinately applied to the common diameter, it may be shewn as above, that the sections would have a fourth common point, contrary to the preceding prop.; and therefore, in every case, two sections, touching each other in two points, do not meet in any other point.

PROP. LXXIII. THEOR.

If a circle touch a conick section or opposite sections in two points; a right line joining the contacts is ordinately applied to an axis of the section.

And if it be ordinately applied to the transverse axis of an ellipse or hyperbola, or to the axis of a parabola, the circle is entirely within the section; but if to the second axis of an ellipse or hyperbola, the circle is entirely without the ellipse, or without both hyperbolas.

Part 1. When the circle touches in two points an ellipse or opposite hyperbolas, if the common tangents be parallel, a right line joining the contacts is a diameter both of the section and the circle (*Cor.* 1. 47. 1 *Sup.*), and since, because of the circle, this diameter is perpendicular to the tangents (18. 3 *Eu.*), it is an axis of the section (*Cor.* 2. 30. 1 *Sup.*); if it be a transverse axis, the circle falls entirely without the ellipse or both hyperbolas, (28 and 29. 1 *Sup.*), and if it be a second axis of an ellipse, the circle falls entirely within the ellipse (28. 1 *Sup*).

Fig. 1.

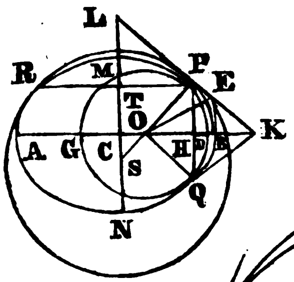
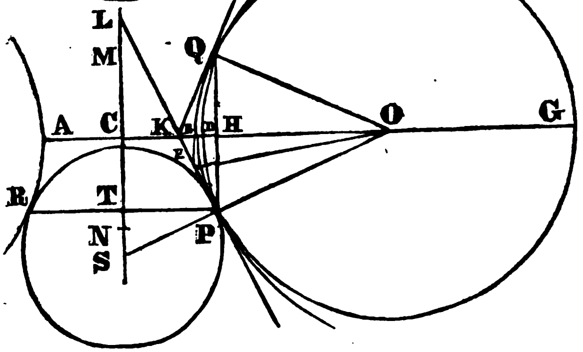


Fig. 2.



But if the circle GPDQ, see both fig. touch the section or or opposite sections in the points P and Q, and the common tangents drawn through these points meet each other in K, PQ being joined is ordinately applied to the axis AB of the section.

For through K let KH be drawn, bisecting PQ in H, and because of the circle, PK and QK are equal (*Cor.* 2. 36. 3 *Eu.*), and KH is common to the two triangles KPH and KQH, these triangles are mutually equilateral, and therefore KH is perpendicular to PQ (8. 1 *Eu.*), but KH is a diameter of the section (49. 1 *Sup.*), and because PQ, which is ordinately applied to it, cuts it at right angles, it is an axis (*Cor.* 2. 30. 1 and *Def.* 12. 1 *Sup.*), whence appears the truth of the first part of the proposition.

Part 2. If the circle GPDQ touch the section in the points P and Q, and PQ be ordinately applied to an axis BH, which in the ellipse and hyperbola is transverse, the circle is entirely within the section.

For BH is perpendicular to PQ (*Cor.* 3. 11. 1 *Sup.*), therefore the centre of this circle is in the axis BH (*by proof of* 1. 3 *Eu.*), course the section, in the point Q (*Cor.* 16. 3 *Eu.* and *Def.* 21. 1 *Sup*).

The proof is similar, when the terminated right line is ordinately applied to the second axis of an ellipse or hyperbola.

PROP. LXXIV. THEOR.

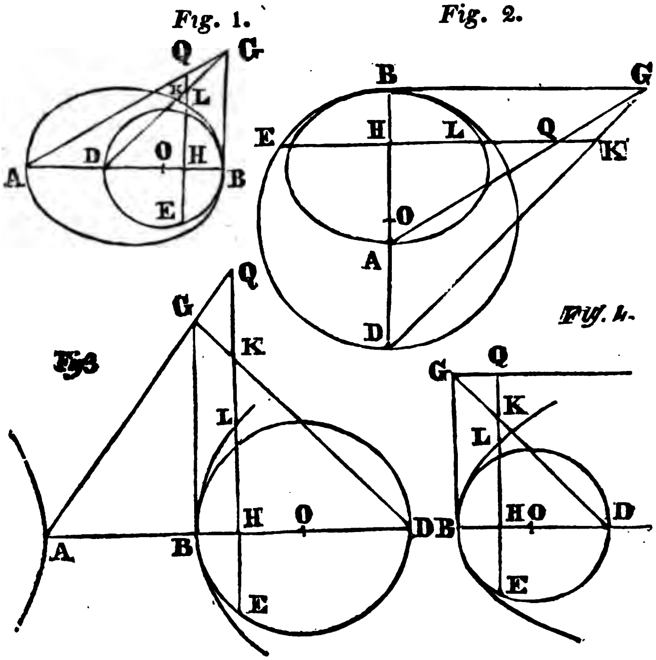
If two circles (QP and RP, see figures to preceding prop.), touch an ellipse or hyperbola in the same point (P), and again touch the same ellipse or opposite hyperbolas in two other points (Q and R); the rectangle contained under the diameters of the circles, is equal to the square of the diameter of the section, which is conjugate to the diameter, passing through the contact of the circles. And if a circle touch a parabola in two points, the square of its diameter, is equal to the rectangle contained under the principal parameter, and the parameter of the diameter of the section passing through the contact.

Part 1. Let AB and MN be the axes of the ellipse or hyperbolas, and C the centre, the right lines PQ and PR joining the contacts are perpendicular to the axes (73. 1 and *Cor.* 3. 11. 1 *Sup.*); through the point P in which both circles touch the section, draw the common tangent (48. 1 *Sup.*), meeting the axes in K and L, and, from P, draw PS perpendicular to KP, meeting the axes in O and S, the points O and S are the centres of the circles (1 and 19. 3 *Eu.*), and, because the diameters of the section passing through the contacts are equal (*Cor.* 3. 11. 1 and 31. 1 *Sup.* and 4. 1 *Eu.*), their conjugates are equal (*Cor.* 1. 53. 1 *Sup.* and 14. 6 *Eu.*]; and the triangles LPS and OPK, being right angled at P, and having the angles at S and K equal, being each the complement of the angle SLK to a right angle, are equiangular, therefore SP is to PL, as PK is to PO (4. 6 *Eu.*), and therefore the rectangle SPO under the semidiameters of the circles, is equal to the rectangle LPK (16 6 *Eu.*), or which is equal (47. 1 *Sup.*), the square of the semidiameter parallel to LP, which semidiameter is conjugate to that passing through P (*Def.* 14. 1 *Sup.*), therefore the rectangle under the diameters of the circles, is equal to the square of the diameter of the section, conjugate to the diameter passing through P (23. 6 and *Ax.* 1. 5 *Eu*).

Part 2. Let PBQ in fig. 2 be a parabola, and let the circle PQ touch it in P and Q, PQ being joined is perpendicular to the axis (73. 1 and *Cor.* 3. 11. 1 *Sup.*), through P draw the common tangent PK (48. 1 *Sup.*), meeting the axis in K, and from P, draw PO perpendicular to the tangent, meeting the axis in O, the point O is the centre of the circle PQ (1 and 19. 3 *Eu.*); and the triangles OPK and OHP, being right angled at P and H. and having the angle KOP common, are equiangular, therefore KO is to OP, as OP is to OH (4. 6 *Eu.*), and therefore the square of OP is equal to the rectangle HOK (17. 6 *Eu.*); but, because the rectangle under BH and the principal parameter, or, KH being double to BH (44. 1 *Sup.*), the rectangle under KH and half that parameter, is equal to the square of PH (23. 1 *Sup.*), or, which is equal (*Cor.* 1. 8. 6 and 17. 6 *Eu.*), the rectangle KHO, therefore HO is equal to half the principal parameter; and, because the rectangle under KB and the parameter of the diameter passing through P, or, KH being double to KB (44. 1 *Sup.*), the rectangle under KH and half the parameter of the diameter passing through P is equal to the square of PK (41. 1 *Sup.*), or, which is equal (*Cor.* 1. 8. 6 and 17. 6 *Eu.*), the rectangle OKH, therefore KO is equal to half the parameter of the diameter passing through P; whence, the square of the semidiameter OP of the circle having been just proved equal to the rectangle HOK, the square of that semidiameter is equal to the rectangle under half the principal parameter, and half the parameter of the diameter of the section passing through P, and therefore the square of the diameter of the circle, is equal to the rectangle under the principal parameter, and the parameter of the diameter of the section passing through P (23. 6 and *Ax.* 1. 5 *Eu*).

PROP. LXXV. THEOR.

If from the vertex of the transverse axis of an ellipse or hyperbola, or the axis of a parabola, there be put in the axis towards the interior of the section, a right line equal to its parameter; a circle described about this, as a diameter, falls entirely within the section; and if from the vertex of the less axis of an ellipse, there be put in the axis, towards the interior of the section, a right line equal to its parameter; a circle described about this as a diameter, falls entirely without the section.



Let AB in fig. 1 and 3, be the transverse axis of an ellipse or hyperbola, or in fig. 2, the second axis of an ellipse, and BH in fig. 4, the axis of a parabola, B being in every case its vertex ; let there be taken on each of these axes, from B, towards the interior of the section, a right line BD, equal to its parameter ; a circle described about BD as a diameter, falls, in the case of fig. 1, 3 and 4, entirely within, and, in that of fig. 2, entirely without the section.

Draw BG perpendicular to the axis, and equal to BD, and join DG ; and, in the ellipse and hyperbola, let AQG be drawn, from the vertex A of the axis AB, but, in the parabola, GQ parallel to the axis ; through any point E in the circle, draw EH parallel to BG, meeting the section in L, and BD, DG and GQ in H, K and Q.

Because BD and BG are equal, DH and HK are equal [4. 6 *Eu.*], and because of the circle, the square of EH is equal to the rectangle BHD [3 and 35. 3 *Eu.*], or BHK ; but, because of the section, the square of HL is equal to the rectangle BHQ, for, in the case of fig. 1, 2 and 3, the rectangle AHB is to the square of HL, as AB is to its parameter BG [*Cor.* 4. 40. 1 *Sup.*]

or [4. 6 *Eu.*], as AH to HQ, or [1. 6 *Eu.*], as the rectangle AHB is to the rectangle BHQ, and therefore the square of HL and rectangle BHQ, to which the rectangle AHB has the same ratio, are equal [9. 5 *Eu.*]; and, in the case of fig. 4, the square of HL is equal to the rectangle under BH and BG or HQ [23. 1 *Sup.*]; but HK is less than HQ, unless when B is a vertex of the second axis of an ellipse, in which case HK is greater than HQ; therefore, in the cases of fig. 1, 3 and 4, the rectangle BHK is less than BHQ, and of course the square of EH less than the square of HL, and the right line EH less than HL, and therefore the circle BED is entirely within the section. In like manner it may be shewn, in the case of the second axis of an ellipse, fig. 2, that the circle is entirely without the section.

Cor. 1. Hence, if from a vertex [B] of an axis [BH] of a conick section, there be put in the axis, towards the interior of the section, a right line [BO] in the case of a principal axis, not greater, and of a less axis of an ellipse, not less than half the parameter [BD] of the same axis; that right line [BO is, in the former case, the least, and, in the latter the greatest, which can be drawn from the extreme [O] remote from the vertex, to the section.

For, in the former case, because the circle described from the centre O does not fall without the circle BED, it is entirely within the section; and since this circle, in the latter case, does not fall within the circle BED, it is entirely without the section; whence the thing proposed is manifest.

Cor. 2. From a point [O see next fig.], in the principal axis [BH] of a conick section, and within the section, whose distance from the nearer vertex [B] is greater than half its parameter, to draw the least right line, which can be drawn from that point to the section, on the same side of the axis.

Fig. 1.

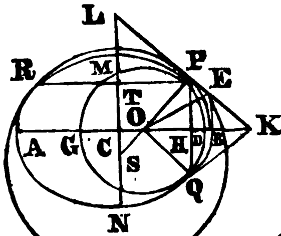
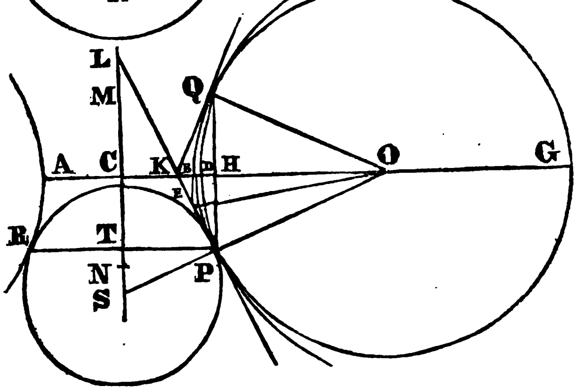


Fig. 2.



If the section be a parabola, let there be put in the axis, towards the vertex B, a right line OH, equal to half the parameter of the axis; and, if the section be an ellipse or hyperbola, let there be put in the axis, from the centre C, such a right line CH, towards the vertex B, that CH may be to OH, as the axis AB is to its parameter [*Cor.* 1. 10. 6 *Eu.*], and, in every section, let there be drawn through H, a right line PHQ perpendicular to the axis, meeting the section in P, OP being joined is the right line required.

For through P, let a tangent to the section PK be drawn [48. 1 *Sup.*], meeting the axis BH in K, then, in the case of a parabola, the square of PH is equal to the rectangle under BH and the principal parameter [23. 1 *Sup.*], or, KH being double of BH [44. 1 *Sup.*], and HO equal to half the parameter, to the rectangle KHO, therefore the angle KPO is a right angle [17. 6 and *Cor.* 2. 8. 6 *Eu.*], and therefore OP the least right line, which can be drawn from O to the section on the same side of BH [*Cor.* 1. 73. 1 *Sup*).

And, in the case of an ellipse or hyperbola, the rectangle AHD, or which is equal [45. 1 *Sup.*], CHK is to the square of PH, as AB to the principal parameter [*Cor.* 4. 4. *Sup.*], or [*Constr.*], as CH is to OH or [1. 6 *Eu.*], as the rectangle CHK is to the rectangle OHK, whence the square of PH and the rectangle OHK, to which the rectangle CHK has the same ratio, are equal [9. 5 *Eu.*], and therefore the angle OPK a right one [17. 6 and *Cor.* 2. 8. 6 *Eu.*], and of course, as in the former case, OP the least right line, which can be drawn from O to the section on the same side of the axis BH [*Cor.* 1. 73. 1. *Sup*].

Cor. 3. And if, in the second axis [MN] of a hyperbola, there be given a point S at any distance from the centre, a right line may in like manner be drawn, the least of any which can be drawn to the same section; namely, by taking CT to ST, as the axis MN is to its parameter, drawing to the section, TP perpendicular to CS, and joining SP. For PL being drawn touching the section in P, and meeting the second axis is L; the square of CM and CT together, or [45. 1 *Sup.*], the rectangle CTL, is to the square of PT, as the second axis MN to its parameter [*Cor.* 4. 40. 1. *Sup.*], or [*Constr.*], as CT is to ST, or [1. 6. *Eu.*], as the rectangle CTL is to the rectangle STL; whence, the square of PT and rectangle STL, to which the rectangle CTL has the same ratio are equal [9. 5 *Eu.*], and the angle LPS a right one [17. 6 and *Cor.* 2. 8. 6 *Eu.*], and therefore SP the least right line, which can be drawn from S to the section BP, on the same side of the axis MN [*Cor.* 1. 73. 1 *Sup*].

Cor. 4. In like manner may be drawn, from any point [S] in the second axis of an ellipse, whose distance [SM] from its more remote vertex is less than half its parameter, the greatest right line which can be drawn from that point to the section on the same side of the axis; namely, by taking CT to ST, as the axis MN is to its parameter, drawing to the section, TP perpendicular to MN, and joining SP. For PL being drawn touching the section in P, and meeting the second axis in L; the rectangle MTN, or [45. 1 *Sup.*], CTL is to the square of PT, as the second axis MN to its parameter [*Cor.* 4. 40. 1 *Sup.*), or [*Constr.*], as CT is to ST, or [1. 6 *Eu.*], as the rectangle CTL is to the rectangle STL; whence the square of PT and rectangle STL, to which the rectangle CTL has the same ratio, are equal [9. 5 *Eu.*], and the angle LPS a right one [17. 6 and *Cor.* 2. 8. 6 *Eu.*], and therefore SP the greatest right line which can be drawn from S to the section on the same side of MN [*Cor.* 1. 73. 1 *Sup*].

PROP. LXXVI. THEOR.

If from a point in a conick section, there be drawn a right line ordinately applied to either axis, and from the point, wherein it again meets the section or opposite sections, a diameter be drawn, and from the first point, a right line be drawn ordinately applied to this diameter; a circle, which touches the section in the first point, and passes through the point, in which the last drawn ordinate again meets the section, has the same curvature as the section, in the point of contact.

Fig. 1.

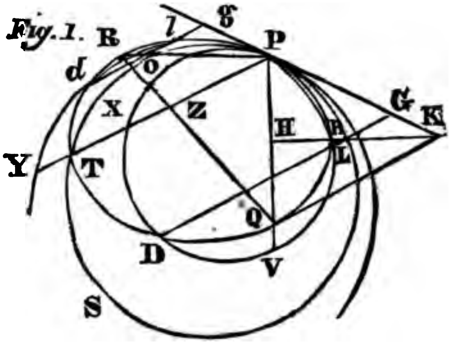


Fig. 2.

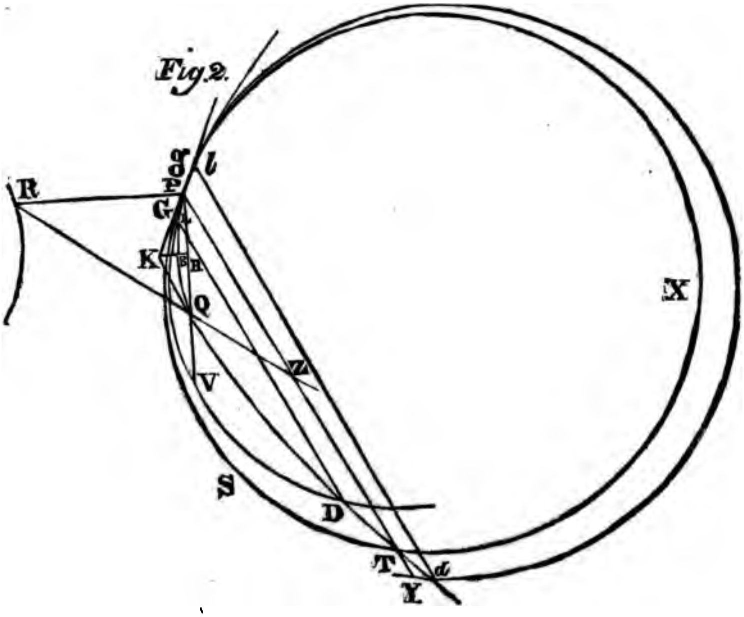
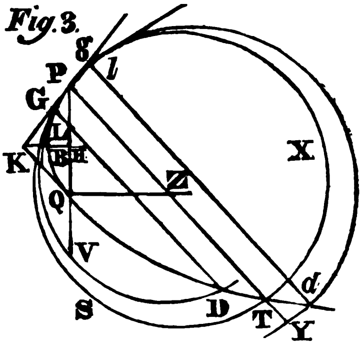


Fig. 3.



From a point P in a conick section, see fig. 1, 2 and 3, representing the respective cases of an ellipse, hyperbola or parabola, let PQ (or PR in fig. 1 and 2), be ordinately applied to an axis, meeting the section again in Q (or R in fig. 1 and 2), and from Q or R let the diameter QZ be drawn, (meeting the section again, in the cases of an ellipse or hyperbola, in R or Q), and from P, let a right line PZT be drawn, ordinately applied to the diameter QZ, and meeting the section again in T; a circle PSTX, touching the section in P, and passing through T, has the same curvature as the section, in the point of contact P.

First, let the axis to which the ordinate is drawn from P be the principal one BH, and through P let the common tangent PK be drawn, meeting the axis BH in K, and through Q, let the right line QK be drawn touching the section (48. 1 *Sup.*), this meets the other tangent PK at K in the same axis (*Cor.* 2. 49. 1 *Sup.*), and because the triangles PHK and QHK have the sides PH and HK and the included angle PHK, severally equal to QH and HK and the included angle QHK, the right lines PK and QK are equal (4. 1 *Eu*).

And first, the circle PXTS does not meet the section unless in the points P and T, and is, on one part of the right line PT, entirely without, and, on the other part, entirely within the section. For it cannot meet it in the point Q, because it would then touch it in that point (*Cor.* 2. 73. 1 *Sup.*), and of course would not meet the section in a third point T (72. 1. *Sup.*), and for the same reason, in the case of fig. 1 and 2, it cannot meet the section in the other vertex R of the diameter QR, for PR being joined is, on account of QP and QR being bisected by the axis BH, parallel to that axis (2. 6 *Eu.*), and of course ordinately applied to the other axis. But, if possible, let this circle meet the section; not only in the points P and T, but also in a third point, either towards S with respect to the point T, as in D, or on the opposite side of T, as in d; in either case, through that third point, let a right line be drawn parallel to QK, meeting PK in G or g, and the section again in L or l; or DG or dg, being parallel to QK, is not a tangent to the section, unless, in the cases of fig. 1 and 2, it be drawn through R (*Constr.* 36. 1 *Sup.*); in which point, it has been just shewn, this circle cannot meet the section. Then, in the case of the circle meeting the section in D, the rectangle DGL is to the square of PG, as the square of QK to the square of PK (14. 1 *Sup.*), or in the ratio of equality, and therefore, because D is in a circle which PK touches in P, the point L is in the same circle, for if the circle meet DG in any other point between D and G except L, the rectangle under the segments of DG between G and the circle, would not be equal to the square of PG, contrary to 36. 3 Eu, therefore the circle PXTS meets the section in three points T, D and L, besides the point of contact P, which is absurd (71. 1 *Sup*). A like absurdity would follow, if the circle were supposed to meet the section in d, as may be in like manner shewn, only using the small letters d, g and l, for their respective capitals. Therefore the circle PXTS does not meet the section, unless in the points P and T; and since it does not touch the section in T, for then PT would be ordinately applied to an axis, by part 1. 73. 1 Sup, contrary to the supposition, the arch of that circle is, on one part of the right line PT, entirely without, and, on the other, entirely within the section.

The same reasoning is applicable to the case, when the axis to which the ordinate PR is drawn, is, in the cases of fig. 1 and 2, the second axis, the diameter QR, to which the other ordinate from P is applied, being the same, as in the other case.

If now any other circle, as PVD, be described, less than PXTS, touching the circle PXTS and the section in P, it would fall within the section on both parts of the point of contact P. For if that circle should pass through Q, it would touch the section there (*Cor.* 2. 73. 1 *Sup.*), and of course would be entirely within the section (73. 1 *Sup.*), as therefore it would be, if the second point in which it met PQ were within the section; but let it meet PQ without the section, as in V; and, because the circle PVD is entirely within the circle PXTS, it necessarily falls, on one part of the right line PT, entirely within the section, namely, on the part X, since the point V on the part S is without the section; and because V is without the section, the less circle being continued on the part X from P must meet the section somewhere between P and V on that part, as in D; through D, let a right line be drawn parallel to QK, meeting the section again in L and PK in G, and it may be shewn as above, that the rectangle DGL is equal to the square of PG, and of course that the point L is in the circle

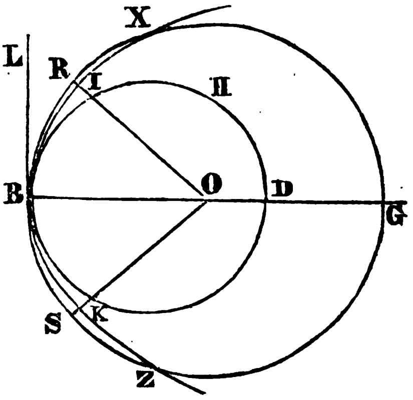
PVD ; but this circle does not touch the section in L, because it touches it in P, and meets it in the points D and L (72. 1 *Sup.*) ; since therefore the arch LV, on one part of the point L, is without the section, the arch LP, on the other part of the point L, is within the section, and is entirely within it, as otherwise the circle PVD would meet the section in another point besides the points P, D and L contrary to prop. 71. 1 *Sup.* Therefore a circle less than PSTX, touching the section in P, falls on both parts of the point of contact P within the section.

If now a circle PdY be described, greater than PXTS, touching the circle PXTS and the section, it would fall without the section, on both parts of the point of contact P. For because it is entirely without the circle PXTS, it necessarily falls on one part of the right line PT, namely, on the part S, entirely without the section ; let this circle PdY, continued from the point P towards the other part X, meet the right line PT produced in Y, it meets the section, in the cases of the hyperbola and parabola, in fig. 2 and 3, when the curves are infinite, somewhere between P and Y on the part X, as in d ; and, in the ellipse, fig. 1, if it does not meet the right line PR, which is ordinately applied to the less axis, within the section, it falls entirely without the ellipse (73. 1 *Sup.*) ; let it then meet PK within the ellipse, as in O ; and since the point Y is without the ellipse, the arch OY meets the ellipse somewhere, as in d ; in every section, let a right line be drawn through d parallel to QK, meeting the section again in l, and PK in g, and it may be shewn as above, that the point l is in the circle PdY, and since this circle touches the section in P, it does not touch it again in the point l or d (72. 1 *Sup.*) ; and, because the point Y is without the section, the arch Yd is without the section, and of course the arch dOl within the same, and therefore the arch lP on the other part of the point l without the section ; and since the circle PdY does not meet the section unless in the points P, l and d (71. 1 *Sup.*), the whole arch lP is without section : Therefore the circle PdY falls on both parts of the point P, without the section.

Since then the circle PXTS, touching the section in the point P, falls on one part of that point, without, and on the other part, within the section ; and any other circle touching the section in P, falls on both parts of that point, either within or without the section, it follows, that no circle can pass between the section and the circle PXTS ; therefore this circle has the same curvature as the section in the point P (*Def.* 22. 1 *Sup*).

PROP. LXXVII. THEOR.

A circle, which touches a conick section, and cuts off from the diameter passing through the point of contact, towards the interior of the section, a segment equal to its parameter, has the same curvature as the section, in the point of contact.



Case 1. Let the diameter passing through the contact be an axis. Let BDG be this axis, B its vertex, and BD a segment taken thereon towards the interior of the section, equal to the parameter of the axis. A circle BHD, described about BD as a diameter, has the same curvature as the section, in the point B.

Let first BGD be the principal axis of a conick section, let the right line BL, drawn through B, touch the circle and section in B (*Hyp.* and *Def.* 21. 1 *Sup.*) ; the circle BHD is entirely within the section (75. 1 *Sup.*), and therefore a circle, touching the section in B, and cutting off from the axis BG, a segment less than BD, falls entirely within the section. But if a circle BRG cut off from the axis, a segment BG greater than its parameter BD, and, in the case of a transverse axis of an ellipse, less than that axis, let its centre be O, and since OB is greater than half the parameter of the axis ; from O let there be drawn to the section the least possible right lines OI and OK on each side of the axis (*Cor.* 2. 75. 1 *Sup.*), which are each of them less than OB (*by the same*) ; therefore a circle described from the centre O at the distance OB would meet OI and OK without the section, as in R and S ; and, because the same circle meets the axis within the section in G, the arch GR necessarily meets the section somewhere between G and R, as in X ; and, in like manner it may be proved that the arch GS meets the section between G and S, as in Z ;

and since this circle does not meet the section, unless in the points B, X and Z (71. 1 *Sup.*), the arches BRX and BSZ are entirely without the section, and therefore the circle BRGS falls on both parts of the point of contact B, without the section; since therefore any circle whatever, touching the section in the point B, whether less or greater than the circle BHD, falls, on both parts of the point of contact B, either within the circle BHD, or without the section, there can no circle pass between the section and the circle BHD; therefore this circle has the same curvature as the section in the point B (*Def.* 22. 1 *Sup*).

The demonstration is entirely the same, when B is the vertex of the less axis of an ellipse, by substituting *greater* for *less*, *greatest* for *least*, *without* for *within*, and the contrary; and citing Cor. 4. 75. 1 Sup, instead of Cor. 2. 75. 1 Sup.

Fig. 1.

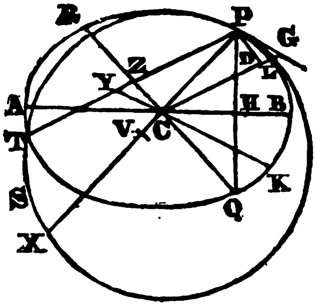
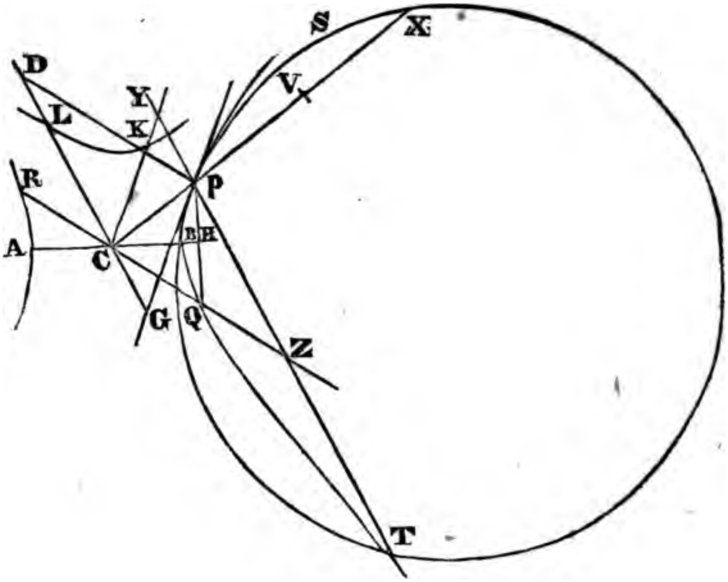


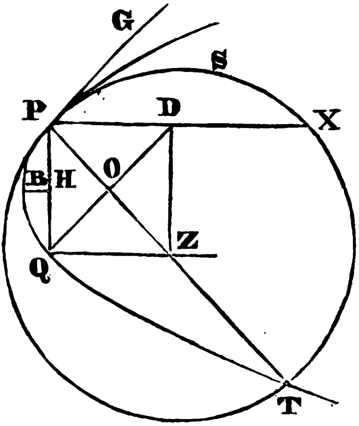
Fig. 2.



Case 2. Let now, in an ellipse or hyperbola, the diameter passing through the contact P, see both fig. not be an axis, let PX be the segment taken thereon, towards the interior of the section, equal to its parameter, and PSX the circle, touching the section in P, and passing through X; the circle PSX has the same curvature as the section, in the point P.

From P, let PH be drawn, ordinately applied to the transverse axis AB, meeting the section again in Q, through Q, let the diameter QCR of the section be drawn, to which let PZ be ordinately applied, meeting the section again in T; let the right line PG be the common tangent of the circle and section, touching both in P (*Hyp.* and *Def.* 21. 1 *Sup.*), and let the diameter CK be drawn parallel to PG, meeting PT in Y, and the diameter CL conjugate to CQ, meeting PG in G, to which let the ordinate PD be drawn, and let PX be bisected in V: and because the semidiameters CP and CQ are equal (4. 1 *Eu.*), their conjugates CK and CL are equal (*Cor.* 1. 53. 1 *Sup.* and 14. 6 *Eu.*), and, because PV is half the parameter PX, the rectangle VPC is equal to the square of CK or CL (*Def.* 15. 1 *Sup.* and 17. 6 *Eu.*), or, which is equal (44. 1 *Sup.* and 17. 6 *Eu.*), the rectangle DCG, or, which is equal (ZP and YP being, because of the parallelograms ZD and YG, equal to CD and CG, by 34. 1 *Eu.*), the rectangle ZPY; whence the rectangles VPC and ZPY being equal, their doubles, the rectangles XPC and TPY are equal (*Ax.* 6. 1 *Eu.*), therefore, YC being parallel to the tangent PG (*Constr.*), the point T is in the circle PSX, for if the circle PSX did not meet the right line PZ in T, the rectangle under the segments of the right line PZ, between P, and the points, in which it meets the circle again, and YC, would not be equal to the rectangle XPC, contrary to Theor. at 5. 4 Eu, therefore the circle PSX has the same curvature as the section, in the point P (76. 1 *Sup*).

Case 3. Let now, in a parabola, the diameter passing through the contact P, not be the axis, PX the segment thereof equal to its parameter, and PSX the circle touching the section in P, and passing through X.



Let there be drawn from P, to the axis BH, a right line PH, ordinately applied to it, meeting the parabola again in Q, and through Q let the diameter QZ be drawn, to which from the point P, let PZ be drawn, ordinately applied to it, meeting the parabola again in T, from Q let the ordinate QD be drawn to the diameter PX, meeting PZ in O.

Because the parameters of the diameters PD and QZ are equal (*Cor.* 4. 11. 1 *Sup.*), and the abscissas PD and QZ are equal (*Cor.* 44. 1 *Sup.*), the ordinates PZ and QD are equal (40. 1 *Sup.* and *Cor.* 1. 46. 1 *Eu.*) ; but, because the triangles POD and QOZ are equiangular, and their sides PD and QZ equal, PO is equal to OZ (4. 6 and 14. 5 *Eu.*) ; whence, PT being bisected in Z (31. 1 *Sup.*), PO is a fourth part of PT, and the rectangle TPO is equal to the square of PZ or QD, or, which is equal (40. 1 *Sup.*), to the rectangle XPD ; whence, QD being parallel to a right line touching the circle and section in P (*Def.* 12. 1 *Sup.*), the point T is in the circle PSX, for otherwise the rectangle under the segments of the right line PZ, between P, and the points, in which it meets the circle again, and QD, would not be equal to the rectangle XPD, contrary to Theor. at 5. 4 Eu, therefore the circle PSX has the same curvature as the section, in the point P (76. 1 *Sup*).

PROP. LXXVIII. THEOR.

If two ellipses have a common axis, two hyperbolas a common transverse axis, or two parabolas a common axis and principal vertex; coincident ordinates to the common axis, are to each other, in a given ratio; being, in ellipses, as the other axes; in hyperbolas, as the respective second axes; and, in parabolas, in a subduplicate ratio of the principal parameters of the parabolas. And the areas of segments cut off by such ordinates subtended by the same segment of the common axis, or, in ellipses, by the whole common axis, are to each other, in the same ratio.

Fig. 1.

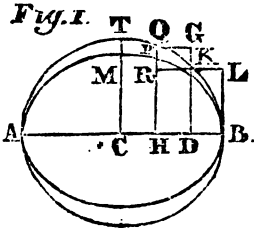


Fig. 2.

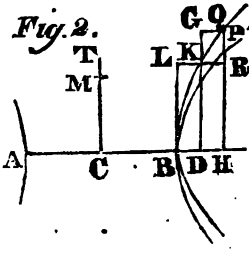
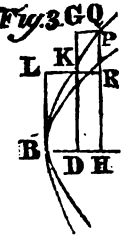


Fig. 3.



Let AB in fig. 1 and 2, be an axis of an ellipse or hyperbola, which in the hyperbola is a transverse one, CM and CT the halves of other axes of the sections BP and BQ; in fig. 3, let BH and B be the common axis and principal vertex of the parabolas BP and BQ, the right lines HP and HQ being in every fig. coincident ordinates to the axis HB; these ordinates are to each other in fig. 1 and 2, as the axes, of which CM and CT are the halves, and in fig. 3, in a subduplicate ratio of the principal parameters of the parabolas.

And the areas HBP and HBQ, and in fig. 1, APB and AQB, are to each other in the same ratio.

Part 1. In fig. 1 and 2, the square of HP is to the rectangle AHB, as the square of CM is to the square CB (23. 1 *Sup.*), and the rectangle AHB is to the square of HQ, as the square of CB to the square of CT (23. 1 *Sup.* and *Theor.* 3. 15. 5 *Eu.*); therefore, by ordinate equality, the square of HP is to the square of HQ, as the square of CM is to the square of CT (22. 5 *Eu.*), and of course HP to HQ, as CM to CT (22. 6 *Eu.*), or as the axes, of which CM and CT are the halves.

In fig. 3, the square of HP is equal to the rectangle under BH and the principal parameter of the parabola BP (23. 1 *Sup.*), and the square of HQ is equal to the rectangle under BH and the principal parameter of the parabola BQ (*by the same*); therefore the square of HP is to the square of HQ, as the principal parameter of the parabola BP is to the principal parameter of the parabola BQ (*Cor.* 1. 7. 5 and 1. 6 *Eu*), and therefore HP is to HQ in a subduplicate ratio of these parameters (*Cor.* 1. 20. 6 *Eu*).

Part 2. Bisect in every fig. HB in D, from D, draw DG parallel to HQ, meeting the section QB in K, and a right line QG drawn through Q, parallel to HB, in G, and from B, draw BL parallel to HQ, meeting a right line drawn through K, parallel to HB in L, and let LK produced meet HQ in R; and there is circumscribed about the figure HQB formed by the section QB and the right lines QH and HB, the rectilineal figure HQGKLB, and inscribed in the same figure HQB, the rectilineal figure HRKD, and the difference between the inscribed and circumscribed figures is the rectangles QK and KB, which together are equal to the rectangle under HD and HQ; and if HD and DB be bisected, and figures be in like manner inscribed in and circumscribed about the figure HQB, it may the same way be proved, that the difference between the inscribed and circumscribed figures, is equal to the rectangle under HQ and the half of HD; and in like manner, if the bisections be continued ever so often, it may be shewn, that the difference between the incribed and circumscribed figures is always equal to the rectangle under HQ and one of the parts into which HB is divided; and since, by repeated bisections, the parts into which HB is divided become at length less than any given right line (*Cor. Theor. at* 7. 5 *Eu.*), therefore the difference between the inscribed and circumscribed figures, and therefore between either and the figure HQB, becomes at length less than any given surface; in like manner, if figures be thus inscribed in and circumscribed about the figure HPB, it may be shewn, that the difference between either the inscribed or circumscribed figures and HPB, becomes at length less than any given surface; and since, the rectangles, as QD, KB, &c. which constitute figures so circumscribed about HQB and HPB, have for their common attitude the parts into which HB is divided, having for the bases of any two corresponding or partly coincident ones, right lines intercepted between HB and the sections PB and QB, which right lines are to each other, as the right lines HP and HQ, (*by part* 1), these rectangles are to each other, in the same ratio (1. 6 *Eu.*), and therefore the sums of all the rectangles, which constitute the circumscribed figures, are to each other, in the same ratio (12. 5 *Eu*); whence the figures so circumscribed about the figure HPB and HQB, by repeated bisections of HB, approaching nearer and nearer to equality with those figures, so as at length to differ from them by magnitudes less than any given ones, and always retaining the same ratio to each other, namely, that of HP to HQ, or, in fig. 1 and 2, of the other axes, or, in fig. 3, a subduplicate one of the principal parameters of the sections, the areas HPB and HQB are to each other in the same ratio (*Theor.* 2. 33. 6 *Eu*). In like manner, in fig. 1, the areas APH and AQH may be proved to be in the same ratio; as are therefore the areas APB and AQB (12. 5 *Eu*).

Cor. 1. Since the same conclusion would follow, if, in the case of fig. 1, one of the figures as AQB were a circle, for, in that case the ratio of the square of HQ to the rectangle AHB would be that of equality (3 and 35. 3 *Eu.*), and therefore equal to that of the square of CT to the square of CB, it follows, that the area of an ellipse is to that of a circle described about the greater axis as a diameter, as the less axis is to the greater, and to the area of a circle so described about the less axis, as the greater axis is to the less.

Cor. 2. In this proposition, from the supposition, that right lines drawn from HB, perpendicularly to the curves PB and QB, are in a constant ratio, it is demonstrated, that the areas HPB and HQB are in the same ratio; therefore, "if, in two

" figures, bounded each by two right lines at right angles to " each other, and a curve, the right lines considered as bases of " each be equal, and all perpendiculars to the equal bases, " intercepted between those equal bases and the curve, be in a " constant ratio, the areas of the figures are in the same ratio."

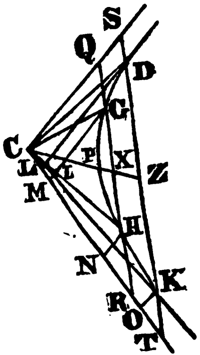
Cor. 3. " And, the same things being supposed, except that " the right lines intercepted between the bases and the curve, " and in a constant ratio to each other, instead of being perpendicular to the bases, form any equal angles whatever with " them towards the curves, the areas of the figures would be " still in the same constant ratio ;" the same reasoning applying except that the parallelograms, instead of being right angled, are oblique angled, equiangular and equilateral parallelograms being equ a by Cor. 3. 34. 1 *Eu.*

Cor. 4. " And if, instead of the right lines, intercepted " between the bases and the curve, and in a constant ratio to " each other, forming equal angles with the bases towards the " curves, in both figures ; the equal angles in one figure, be " complements of the equal angles in the other, to two right " angles, the areas of the figures would be still in the same " constant ratio ;" for since the acute angles of oblique angled parallelograms, are complements of their obtuse ones to two right angles (29. 1 *Eu.*), the parallelograms used in the proof would be still equiangular, and therefore the same reasoning would in this case also apply.

PROP. LXXIX. THEOR.

A hyperbolick sector (CGH), formed by a hyperbolick arch (GH), and two semidiameters (CG and CH) drawn to its extremes (G and H), is equal to the hyperbolick trapezium (MGHN) formed by the same arch, by two right lines (GM and HN), drawn from its extremes (G and H), to one asymptote (CT), parallel to the other (CS), and the segment (MN) of the asymptote, to which the parallels are drawn, between those parallels.

And any two such sectors (CDG and CHK), or trapeziums (LDGM and NHKO), of which the segments (CL and CM, CN and CO) of the asymptote, to which parallels are so drawn from the extremes of the hyperbolick arches, between those parallels, and the centre, are in the same ratio to each other in both, are equal.



Part 1. The triangles CMG and CNH are equal (*Cor.* 2. 38. 1 *Sup.* and 16 and 15. 6 *Eu.*), taking from each the common triangle CMI, the triangle CIG is equal to the trapezium MIHN, to each of which adding the space IGH, the sector CGH is equal to the hyperbolick trapezium MGHN.

Part 2. Let now right lines DL, GM, HN and KO drawn from the extremes of the hyperbolick arches DG and HK parallel to one asymptote CS, meet the other CT in L, M, N and O, and CL be to CM, as CN to CO; the sectors CDG and CHK are equal.

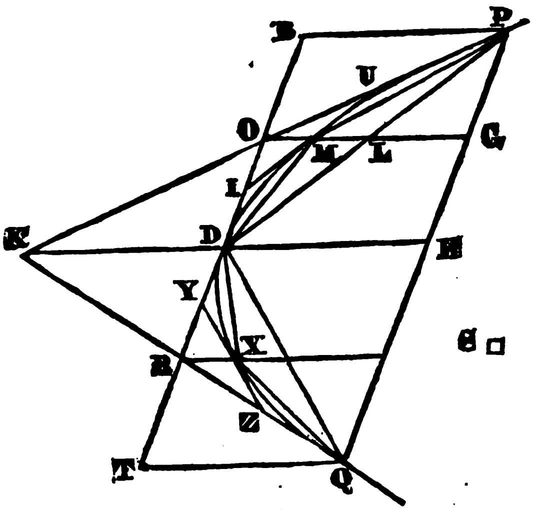
For, drawing the right lines GH and DK, and producing them to meet the asymptotes in Q and R, S and T, because NH, MG and CQ are parallel, NR is to HR as CM to QG (*Cor.* 2. 10. 6 and 16. 5 *Eu.*), therefore because of the equals QG and HR (37. 1 *Sup.*), the right lines CM and NR are equal (*Theor.* 3. 15. 5 and 14. 5 *Eu.*); and in like manner, because of the parallels OK, LD and CS, and the equals SD and KT, the right lines CL and OT are equal; whence NR is to OT, as CM is to CL (*Cor.* 1. 7. 6 *Eu.*), or, which is equal, (*Hyp.*), as CO is to CN, or, which is equal (*Cor.* 2. 38. 1 *Sup.* and 16. 6 *Eu.*), as NH is to OK; whence, the angles RNH and TOK being equal (*Hyp.* and 29. 1 *Eu.*), the angles NRH and OTK are equal (6. 6 *Eu.*), and therefore the right lines RHQ and TKS are parallel (28. 1 *Eu*). Let then the semidiameter be drawn bisecting DK in Z, and meeting the hyperbola and GH in P and X, and because PZ bisects GH and all other right lines terminated by the hyperbola and parallel to DK (*Cor.* 1. 32. 1 *Sup.*), and the angles formed by these right lines with PZ towards D, are complements of the angles formed by the same right lines with PZ towards K to two right angles (13. 1 *Eu.*), the areas PDZ and PKZ are equal (*Cor.* 4. 78. 1 *Sup.*); for a like reason, the areas PGX and PHX are equal, and therefore their differences, the areas XZDG and XZKH are equal, which with the triangles CXG and CXH, which are equal (38. 1 *Eu.*), being taken from the triangles CZD and CZK, which are also equal (*by the same*), the residues, namely the hyperlolick sectors CGD and CHK, are equal.

And since the hyperbolick trapeziums LDGM and NHKO are severally equal to the equal hyperbolick sectors CGD and CHK (*by part* 1), they are equal to each other.

Cor. Hence, if in an asymptote CT of a hyperbola, there be taken from the centre C any number of parts CL, CM, CN, &c. continually proportional, and right lines LD, MG, NH, &c. be drawn parallel to the other asymptote CS; the sectors CDG, CGH, &c. as also the trapeziums LDGM, MGHN, &c. are all equal to each other.

PROP. LXXX. THEOR.

If a triangle (PKQ) being formed by a right line (PQ) inscribed in a parabola, and tangents (PK and QK) to the figure, drawn from its extremes, the inscribed right line be bisected (as in H), and its parts be again bisected, and so on; and diameters being drawn through all the bisecting points G, H, &c. an inscribed figure be formed by joining the extremes (P and Q), of the inscribed right line, and the vertices (M, D, &c.) of these diameters by right lines, and a circumscribed figure, by drawing tangents through these vertices; the inscribed area, is always double the area, contained between the circumscribed figure, and the same triangle.



First, let the inscribed figure be the triangle PDQ, O and R being the points in which the tangent through D meets PK and QK, the circumscribed figure is the trapezium PORQ, and the area without it, the triangle OKR; and, because HD is a diameter, it passes through the point K (*Cor.* 2. 49. 1 *Sup.*), and because KH is double to KD (44. 1 *Sup.*), the triangles KDO and KHP being equiangular, HP is double to DO (4. 6 and 16. 5 *Eu.*); for a like reason, QH is double to RD, therefore QP is double to RO, and the triangle QDP double to RKO (1. 6 *Eu*).

Again, if PH be bisected in G, and through G, the right line OMG be drawn parallel to KH, meeting PD in L, because of the parallels MG and DH, the diameter MG bisects PD in L (2. 6 *Eu.*), and therefore passes through O (*Cor.* 2. 49. 1 *Sup.*); whence, because of the equals LM and MO, the triangle DMP is double the triangle IOU (1. 6 *Eu.*); in like manner, the triangle DXQ may be proved to be double to YRZ, but DMP and DXQ are added to the former inscribed figure, and IOU and YRZ are added to the area which is without the circumscribed figure; therefore the inscribed figure PMDXQ is double the figure UIYZK between the circumscribed figure and the triangle PKQ; and in like manner, if the number of the sides of the inscribed and circumscribed figures, be ever so much increased, whatever is added to the area without the circumscribed figure, its double is always added to the inscribed figure, and therefore the inscribed figure is always double the area, which is without the circumscribed figure, and within the triangle PKQ.

Cor. 1. Hence it is manifest, that no figure can be incribed in the parabolick segment PDQ, as in this proposition, which is double the area PKQD without the parabolick segment and within the triangle PKQ.

Cor. 2. It is also manifest, that no figure can be circumscribed about the parabolick segment PDQ, as in the proposition, so that the area without it, and within the triangle PKQ would be half the parabolick segment PDQ.

Cor. 3. In the parabolick segment PDQ, a figure may be inscribed as in this proposition, which would want of it by a space less than any given one.

For the triangle DHQ is half of the triangle KHP, and DHQ of KHQ (1. 6 *Eu.*), and therefore PDQ of PKQ, therefore the triangle PDQ is more than half the parabolick segment PDQ; and, in like manner, if, in the remaining segments, the triangles PMD and DXQ be inscribed, they would take away more than half of these segments, and if, in the remaining segments, this be continually done, the same would always happen; wherefore the inscribed figure would at length want of the parabolick segment by a space less than any given one (*Theor. at* 7. 5 *Eu*).

Cor. 4. About the parabolick segment PDQ, a figure may be circumscribed, as in this proposition, which would take from the area PKQD, without the parabolick segment PDQ, and within the triangle PKQ, a space less than any given one.

In the area PKQD, let the triangle OKR be inscribed, whose base OR is parallel to the right line PQ joining the contacts, and because KD and DH are equal (44. 1 *Sup.*), KR and RQ as also KO and OP are equal (2. 6 *Eu.*); therefore the triangle OKR is half the triangles KDP and KDQ together, (38. 1 *Eu.*), and of course more than half the area PKQD; and, in like manner, if in the remaining segments PODM, and QRDX, the triangles OIU and RYZ be inscribed, they would take away more than the halves of these segments; and if this be continually done, the same would always happen; therefore the circumscribed figure would at length take away from the area PKQD, a space less than any given one (*Theor. at* 7. 5 *Eu*).

PROP. LXXXI. THEOR.

A parabolick segment (PDQ, see fig. to prec. prop.) formed by a parabolick arch, and a right line joining its extremes, is two thirds of a circumscribing parallelogram (PBTQ), formed by its base (PQ), a tangent (BT) to the parabola parallel thereto, and diameters (PB and QT) drawn from its extremes (P and Q).

Let the tangents PK and QK be drawn (48. 1 *Sup.*), and through their concourse K. a diameter KH, which bisects PQ in H (*Cor.* 1. 49. 1 *Sup.*), and through its vertex D, a tangent ODR, the diameter KH passes through the contact D (32. 1 and *Def.* 12. 1 *Sup.*), and because of KH double to DH (44. 1 *Sup.*), the triangle PKQ is equal to the parallelogram PBTQ; but the parabolick segment PDQ is double the area PKQD, for if it exceeded the double of that area, by any space S, then, by Cor. 3 of the preceding, a figure may be inscribed in the segment PDQ, which would want of that segment by a space less than S, therefore this figure would be greater than double the area PKQD, contrary to Cor. 1 of the preceding, therefore the segment PDQ does not exceed double the area PKQD.

Neither is it less than double this area; for if it were, and therefore the area PKQD were greater than half the segment PDQ, by any space S, then, by Cor. 4 of the preceding, a figure may be circumscribed about this segment, within the triangle PKQ, which would take away from the area PKQD, a space less than S; therefore the area without this figure, and within the triangle PKQ, would exceed half the segment PDQ, contrary to *Cor.* 2 of the preceding; since then the segment PDQ is neither greater nor less than double the area PKQD, it is double that area, and is therefore to the triangle PKQ, or its equal, the parallelogram PBTQ, as two is to three, and is of course two thirds of that parallelogram.

Cor. Since the parabolick areas DBP and DTQ are together one third part of the parallelogram BQ, and the triangle BPD is one half of the parallelogram BH; it follows, that the parabolick area BPD is two thirds, and the segment DMP one third of the triangle BPD. See Newt. Princip. Cor. 5. Lem. 11th Book I.

PROP. LXXXII. THEOR.

If a right line (GD, see both fig.), meeting a conick section or opposite sections in two points (S and T), meet two tangents to the section (PK and QK), and the right line (PQ) joining the contacts; the rectangles (SDT and SGT), under the segments of the secant, between the tangents, and the section, are to each other, as the squares of its segments (DH and GH), between the tangents, and the right line joining the contacts.

Fig. 1.

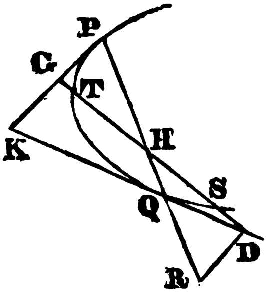
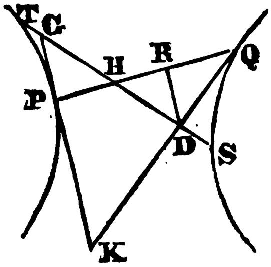


Fig. 2.

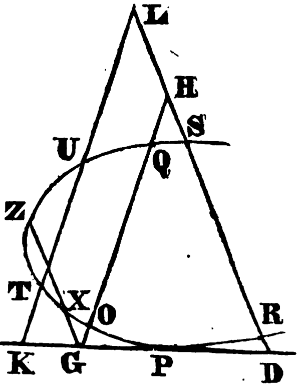


Through the point D, in which the secant meets one of the tangents KQ, let a right line DR be drawn, parallel to the other tangent PK, meeting the right line joining the contacts in R; then is the rectangle SDT to the square of DR, as the rectangle SGT to the square of PG (65. 1 *Sup.*); therefore, by alternating, the rectangle SDT is to the rectangle SGT, as the square of DR is to the square of PG (16. 5 *Eu.*), or, (because of the equiangular triangles HDR and HGP), as the square of GH is to the square of DH (4 and 22. 6 and 16. 5 *Eu*).

If the tangents, which the secant meets, be parallel to each other, the proposition is manifest from Cor. 5. 14. 1 Sup.

PROP. LXXXIII. THEOR.

If two right lines (DH and GH, or DL and KL), meeting each other, and cutting a conick section or opposite sections in two points, meet a tangent (KD) to the same; the squares of the segments (GP and DP, or KP and DP) of the tangent, between its concourses with the secants, and the contact, are to each other, in a ratio compounded of the direct ratio of the rectangles (OGQ and RDS, or TKU and RDS), under the segments of the secants, between the tangent and section, and the inverse one of the rectangles (QHO and SHR, or ULT and SLR) under the segments of these secants, betweeen their concourse and the section.



Case 1. When a right line as GZ, drawn through the point G, in which one of the secants meets the tangent, parallel to the other secant DH, meets the section, as in X and Z.

Having drawn GZ parallel to DH, because of the parallel secants GZ and DL, the square of GP is to the square of PD, as the rectangle XGZ is to the rectangle RDS (*Cor.* 1. 14. 1 *Sup.*); but the ratio of the rectangle QGZ to the rectangle RDS is compounded of the ratio of the rectangle XGZ to the rectangle SHR, or, (*Cor.* 4. 14. 1 *Sup.*), of the ratio of the rectangle OGQ to the rectangle QHO, and of the ratio of the same SHR to RDS (*Def.* 13. 5 *Eu.*); therefore the ratio of the square of GP to the square of PD is compounded of the ratio of the rectangle OGQ to QHO, and the ratio of the rectangle SHR to RDS, or, which is equal (*Cor.* 5. 23. 6 *Eu.*), of the ratio of the rectangle OGQ to RDS, and the ratio of SHR to QHO.

Case 2. When the secants KL and DL and the tangent KD are so posited, that a right line drawn from the point K, in which one of the secants KL meets the tangent, parallel to the other secant DL, would not meet the section.

Through any point X in the section between P and T, draw GZ parallel to DL, meeting PK in G, and the section again in Z; the ratio of the square of KP to the square of PD, is compounded of the ratios of the square of KP to the square of

GP, and of the square of GP to the square of PD (*Def.* 13. 5 *Eu.*) ; but the square of KP is to the square of GP, as the rectangle TKU is to the rectangle OGQ (*Cor.* 1. 14. 1 *Sup.*), and the square of GP is to the square PD, in a ratio compounded of the ratios of OGQ to RDS and of SHR to QHO (*by case* 1) ; therefore the square of KP is to the square of PD, in a ratio compounded of the ratios of the rectangle TKU to OGQ, of OGQ to RDS and of SHR to QHO ; and the ratio compounded of the ratios of TKU to OGQ and OGQ to RDS, is equal to the ratio of TKU to RDS (*Def* 13. 5 *Eu.*), and the ratio of SHR to QHO to that of SLR to ULT (14. 1 *Sup*) ; therefore the square of KP is to the square of PD, in a ratio, compounded of the ratios of the rectangle TKU to RDS and of SLR to ULT.

PROP. LXXXIV. THEOR.

The diagonals of any quadrangle (ABCD, see fig. 1 *and* 2*), formed by four right lines, touching a conick section or opposite sections, intersect each other, in the same point, as do right lines (EG and HF) joining the opposite contacts.*

Fig. 1.

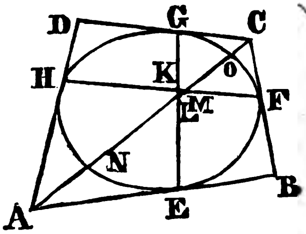
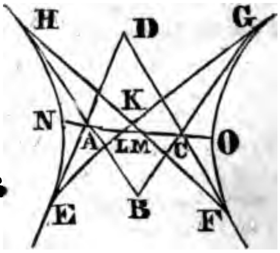


Fig. 2.



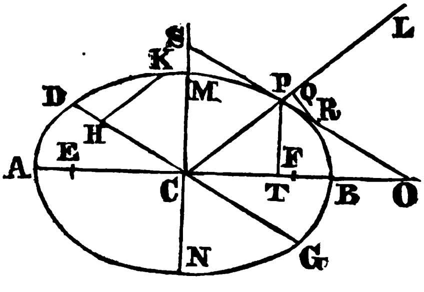
Let K be the intersection of the right lines EG and HF, joining the opposite contacts; and if the diagonals of the quadrangle ABCD, do not intersect each other in K, let one of them, as AC, if possible, meet the right lines EG and HF elsewhere, as in L and M, and let N and O be the points, in which AC meets the section.

Because AC meets the tangents EA and GC in A and C, and the right line EG joining their contacts in L, the square of AL is to the square of LC, as the rectangle NAO is to the rectangle OCN (82. 1 *Sup.*); and because AC meets the tangents HA and FC in A and C, and the right line HF joining their contacts in M, the square of AM is to the square of MC, as the rectangle NAO is to the rectangle OCN (*by the same*); whence, the ratios of the squares of AL and LC and of the squares of AM and MC, being each equal to the ratio of the rectangle NAO to the rectangle OCN, are equal to each other (11. 5 *Eu*), and therefore AL is to LC as AM to MC (22. 6 *Eu.*), which is absurd (8. 5 *Eu.*): therefore a right line drawn from A to C passes through the point K.

In like manner it may be proved, that a right line drawn from D to B passes through the point K: therefore the diagonals of the quadrangle ABCD intersect each other in K.

PROP. LXXXV. PROB.

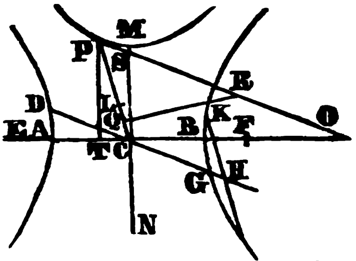
To describe a conick section, of which a diameter (DG), its vertices, or if it have but one, that one, and an ordinate (HK) to the same diameter, are given.



Case 1. Let D and G be the vertices of the diameter DG, and HK meet DG between D and G, the section being of course an ellipse.

Bisect DG in C, and draw CP parallel to HK, taking CP so, that its square may be to the square of HK, as the square of CG is to the rectangle DHG (*Cor.* 3. 23. 6); P is in the ellipse (*Cor.* 3. 40. 1 *Sup.*); through P, draw PO parallel to CG, on CP produced, take PL a third proportional to CP and CG (11. 6 *Eu.*), so is the rectangle CPL equal to the square of CG (17. 6 *Eu.*); bisect CL in Q, and draw QR perpendicular to CL, meeting PO in R; from the centre R, describe a circle through C and L, meeting PO in the points S and O; CG and CS being joined are the axes.

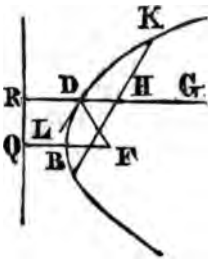
For SO touches the section of which CP and CG are conjugate diameters (*Def.* 14. 1 *Sup.*), and, because of the circle, the rectangle SPO is equal to the rectangle CPL (35. 3 *Eu.*), or to the square of CG, therefore CS and CO are conjugate diameters (*Cor.* 2. 47. 1 *Sup.*), and, because SO is a diameter of the circle, the angle SCO is a right one (31. 3 *Eu.*), and therefore the conjugate diameters CS and CO are the axes (*Def.* 14. 1 and *Cor.* 2. 30. 1 *Sup.*); from P draw PT at right angles to CO, and take CB a mean proportional between CT and CO (13. 6 *Eu.*), and CA equal to CB; A and B are the vertices of the axis, CO (44. 1 *Sup.*); and by letting fall a perpendicular from P on CS, and proceeding in like manner, the vertices M and N of the other axis may be found; divide AB in E and F, so that the rectangles AEB and AFB may be each equal to the square of CM (*Cor.* 2. 6. 2 *Eu.*), the points E and F are the focuses (2. 1 *Sup.*), which being found, describe the ellipse (*Post.* 1. 1 *Sup*).



Case 2. Let now, the points D and G being the vertices of the diameter DG, the ordinate HK meet DG produced, the section being of course a hyperbola

As in preceding case, bisect DG in C, take CP parallel to HK, and whose square may be to the square of HK, as the square of CG is to the rectangle DHG (*Cor.* 3. 23. 6 *Eu.*), P is a vertex of the diameter parallel to HK (40. 1 *Sup.*), and therefore in a hyperbola conjugate to that to be described; through P draw PO parallel to CG, on CP from P towards C take PL a third proportional to CP and CG (11. 6 *Eu.*), so is the rectangle CPL equal to the square of CG (17. 6 *Eu.*); bisect CL in Q, and draw QR perpendicular to CL, meeting PO in R; from the centre R, describe a circle through C and L, meeting PO in S and O; CS and CO being drawn are the axes.

For PO touches the section, of which CP and CG are conjugate diameters (*Def.* 14. 1 *Sup.*), and, because of the circle, the rectangle SPO is equal *to* the rectangle CPL (36. 3 *Eu.*), or, to the square of CG; therefore CS and CO are conjugate diameters (*Cor.* 2. 47. 1 *Sup.*), and, because SO is a diameter of the circle, the angle SCO is a right one (31. 3 *Eu.*), and therefore the conjugate diameters CS and CO are the axes (*Def.* 14. 1 and *Cor.* 2. 30. 1 *Sup.*); from P, draw PT at right angles to AB, and take CA and CB, each a mean proportional between CO and CT; A and B are the vertices of the axis CO (44. 1 *Sup.*); in like manner the vertices M and N of the other axis may be found. In AB, produced both ways, take two points E and F, so that the rectangles AEB and AFB may be each equal to the square of CM (*Cor.* 3. 6. 2 *Eu.*), the points E and F are the focuses (2. 1 *Sup.*), which being found, describe the hyperbola or opposite hyperbolas (*Post.* 2. 1 *Sup*).

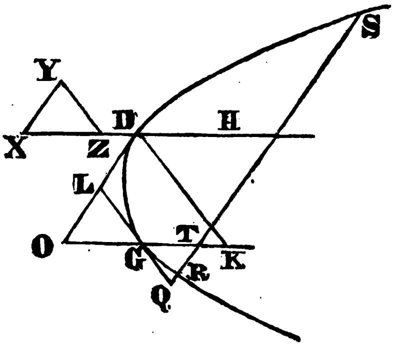


Case 3. Let the diameter DG, the section being a parabola, have but one vertex D; take DR on the part of DG opposite to H, equal to a fourth part of a third proportional to DH and HK (11 and 9. 6 *Eu.*), through D, draw DL parallel to HK, make the angle LDF equal to LDR, and DF equal to DR, and through R, draw RQ perpendicular to DR; describe a parabola from the focus F, with the directrix RQ (*Post.* 3. 1 *Sup.*), which is the required section.

For the parameter of its diameter RDG is equal to four times DR or DF (*Def.* 16. 1 *Sup.*), and the right line DL touches it in D (10. 1 *Sup.*); therefore HK being parallel to DL is ordinately applied to the diameter DG (*Def.* 12. 1 *Sup.*); and since the square of HK is equal to the rectangle under DH and four times DR or the parameter of the diameter DG (*Constr.* and 17. 6 *Eu.*), the point K is in the parabola described (40. 1 *Sup.*), and therefore what was required is done.

PROP. LXXXVI. THEOR.

If a side of any triangle be parallel to the diameters of a parabola, the squares of the other sides are to each other, as the parameters of the diameters, whose ordinates are parallel to those sides.



Let XYZ be a triangle, whose side XZ is parallel to the diameters of the parabola DG, and let DH and GK be the diameters, whose ordinates are parallel to XY and YZ; the square of XY is to the square of YZ, as the parameter of the diameter DH is to the parameter of the diameter GK.

Through D and G the vertices of the diameters DH and GK, draw right lines DL and GL touching the parabola (48. 1 *Sup.*), they are parallel to XY and YZ (*Hyp.* and *Def.* 12. 1 *Sup.*); let DK be drawn ordinately applied to the diameter GK (36. 1 *Sup.*), and DL and GK produced, meet in O ; GK is equal to GO (44. 1 *Sup.*), and therefore, because of the parallels DK and LG, the right line DL is equal to LO (2. 6 *Eu.*) ; but since the triangles XYZ and OLG, having their sides mutually parallel, are equiangular (*Cor.* 3. 9. 1 *Sup.*), the square of XY is to the square of YZ, as the square of OL, or its equal DL, to the square of LG (4. 6 *Eu.*), or, which is equal (42. 1 *Sup.*), as the parameter of the diameter DH to that of the diameter GK.

Cor. 1. Since it appears from this proposition, that the squares of the sides XY and YZ are to each other, as the squares of the segments DL and LG of the tangents parallel to them, between their concourse L, and the contacts D and G, therefore, by 14. 1 Sup. and 9. 5 Eu. the squares of the sides XY and YZ are to each other, as the squares of the segments of tangents, or rectangles under the segments of secants, parallel to them, between their concourse and the section.

Cor. 2. If a right line (GQ) touch a parabola, and from a point (Q) in the tangent, a right line (QS) be drawn, meeting the diameter (GT) drawn through the contact, and cutting the parabola in two points ; the square of the segment (QT) of the secant, between the tangent and the diameter, is equal to the rectangle (RQS) under the segments of the secant, between the tangent and the section.

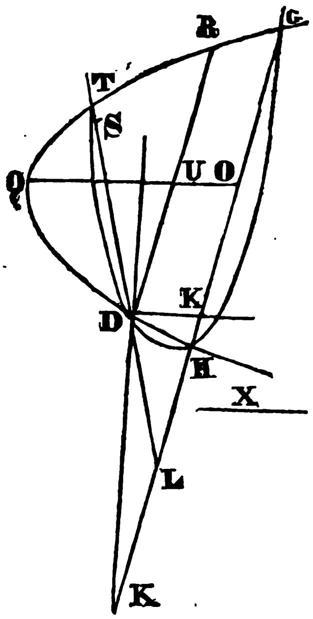
For, by the preceding corollary, the square of QT is to the square of QG, as the rectangle RQS is to the same square of QG ; therefore the square of QT and rectangle RQS are equal (9. 5 *Eu*).

PROP. LXXXVII. PROB.

Through three given points, which are not in the same right line, to describe a parabola, having its diameters parallel to a right line given by position, which is not parallel to a right line joining two of the given points.

Let D, G and H be the three given points, and X the right line given by position; join GH, and through D, draw DK parallel to X; then if GH be bisected in K, it is ordinately applied to the diameter DK (32. 1 *Sup.*), and the problem is performed as in case 3. 85. 1 Sup.; if not, bisect GH in O, through O draw OQ parallel to X, and on the part of GH, to which the point D is, if K be between G and H, and on the contrary part, if not, take OQ to DK, as the rectangle GOH, or square of GO is to the rectangle GKH (*Cor.* 2. 23. 6 *Eu.*); with the diameter QO, vertex Q, and ordinate GO, describe a parabola (85. 1 *Sup.*), and the thing required is done.

For, because DK is to QO, as the rectangle GKH is to the rectangle GOH, the point D is in the described parabola (21. 1 *Sup.*), and its diameters QO and DK are by construction parallel to X.



PROP. LXXXVIII. PROB.

Four points in a parabola being given, to describe it.

Case 1. Let D, H, G and T be the four given points, and first, let the four right lines joining these points in continuation form a trapezium of which no two sides are parallel to each other, let two opposite sides GH and TD meet each other in L, and in the right line LG take LK, so that its square may be to the square of LD, as the rectangle HLG is to the rectangle DLT (*Cor.* 3. 23. 6 *Eu.*), and, having drawn DK, describe a parabola through the points D, H and G, whose diameters are parallel to DK (87. 1 *Sup.*); this passes through the point T.

For since the rectangle DLT is to the rectangle HLG, as the square of DL is to the square of LK (*Constr.* and *Theor.* 3. 15. 5 *Eu.*), and the rectangle DLT is greater than the square of

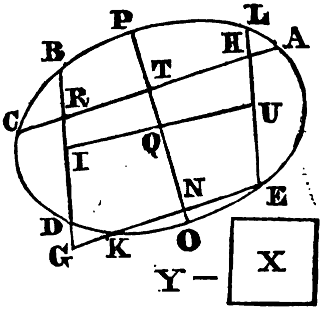
DL (3. 2 *Eu.*), the rectangle HLG is greater than the square of LK (14. 5 *Eu.*), therefore LD does not touch the section, for if it did, the square of LK would be equal to the rectangle HLG (*Cor.* 2. 86. 1 *Sup.*), contrary to what has been just proved, therefore it cuts it in another point, and if not in T, let it, if possible, do it in S, and since the rectangle DLS is to the rectangle HLG, as the square of LD is to the square of LK (*Cor.* 1. 86. 1 *Sup.*), or, which is equal (*Constr.*), as the rectangle DLT is to the rectangle HLG, the rectangles DLS and DLT, having the same ratio to the rectangle HLG, are equal (9. 5 *Eu.*), and SL and TL equal, which is absurd (*Ax.* 9. 1 *Eu.*), the point S being to the same part of L, as the points D and T are, because L is without the parabola; therefore the parabola meets the right line LD in the point T. But because the segment LK may be taken in the right line LHG, on either part of the point L, two parabolas may be described, which would satisfy the problem.

Case 2. Let D, H, G and R be the four given points, and let HG and DR, being joined, be parallel; a right line UO bisecting these parallels is a diameter of the parabola passing by these four points (*Cor.* 2. 32. 1 *Sup.*); let a parabola be described through the points G, H and D, whose diameters are parallel to UO (87. 1 *Sup.*), and since HG is ordinately applied to the diameter UO (32. 1 *Sup.*), DU which is parallel to HG is an ordinate to the same diameter (*Def.* 12. 1 *Sup.*), whence RU being equal to DU, the point R is in the parabola (31. 1 *Sup.*), which therefore passes through this point. It is manifest, that, in this case, there is but one position of the diameters of the parabola which passes through these points, and therefore, that only one can do so.

Cor. From the construction in this proposition, appears a method, of finding the position of the diameters of a parabola, from four points being given in it.

PROP. LXXXIX. PROB.

Five points, (A, B, C, D and E) in a conick section being given, to describe it.



Join AC and BD intersecting each other in R, and through the fifth point E, draw EG and EL parallel to AC and BD, meeting them in G and H. Take Y a fourth proportional to GE, GB and GD (12. 6 *Eu.*), and on GE, take GK, having to Y, the ratio of the rectangles CRA and BRD to each other (*Cor.* 2. 23. 6 *Eu.*), and the rectangle EGK has to the rectangle under Y and GE, or, which is equal (*Constr.* and 16. 6 *Eu.*), the rectangle BGD, the same ratio, as GK has to Y (1. 6 *Eu.*), or which is equal (*Constr.*), as the rectangle CRA has to BRD ; in like manner take the point L, so that LHE may be to CHA, as BRD is to CRA ; but the points E and K, or E and L ought to be on the same or different parts of the points G and H, according as the points B and D, or A and C are on the same or different parts of the same points G and H.

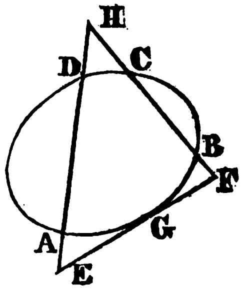
It is manifest from 14. 1 Sup, that the points K and L are in the section passing through the points A, B, C, D and E ; let then a right line IU be drawn, bisecting the right lines BD and LE ; IU is a diameter of the section (*Cor.* 2. 32. 1 *Sup.*) ; let another diameter TN be drawn bisecting the parallels CA and KE in T and N ; if these diameters be parallel, the section is a parabola, in which four points A, B, C and D being given, let it be described through these points (88. 1 *Sup.*), and what was required is done.

But if the diameters IU and TN meet each other, as in Q, the section is an ellipse or hyperbola, whose centre is Q. Let TA be the greater of the two TA and NE, and take a space X, which has the same ratio to the difference of the squares of QN and QT, as the square of TA has to the excess of the square of TA above the square of NE (*Cor.* 2. 47. 1 and *Cor.* 3. 23. 6 *Eu.*) ; take the points O and P in TN produced if necessary, so that the square of QO or QP may, in the ellipse, be equal to the sum, and in the hyperbola, to the difference of X and the square of QT (*Cor.* 1 and 2. 47. 1 *Eu.*), and X is in both cases equal to the difference of the squares of QP and QT, the semidiameter QP being greater than QT in the ellipse, and less in the hyperbola. An ellipse or hyperbola, as the case may be, described with the diameter PO, its vertices being P and O, and the ordinate TA or NE (85. 1 *Sup.*), is the section required.

For since X, or its equal, the difference of the squares of QP and QT, is to the difference of the squares of QN and QT (QT belonging to the greater ordinate TA, being less than QN in the ellipse, and greater in the hyperbola, as is manifest from the constant ratio of the squares of the ordinates to the rectangles under the abscissas), as the square of TA is to the excess of the square of TA above that of NE (*Constr.*), by converting, or comparing the antecedents with the excesses of the antecedents above the consequents, the difference of the squares of QP and QT, or which is equal (*Schol.* 6. 2 *Eu.*), the rectangle PTO, is to the difference of the squares of QP and QN, or (*Schol.* 6. 2 *Eu.*), the rectangle PNO, as the square of TA is to the excess of the square of TA above the difference of the squares of TA and NE, or the square of NE (*Schol.* 18. 5 *Eu.*), since then the squares of TA and NE have the same ratio to each other, as the rectangles PTO and PNO, a conick section described with the diameter PO and ordinate TA, would pass through E, or with the ordinate NE, would pass through A (*Cor.* 3. 40. 1 *Sup*).

PROP. XC. PROB.

Four points (A, B, C and D) in a conick section being given, and a right line (EF) touching it, being given by position, to describe the section.

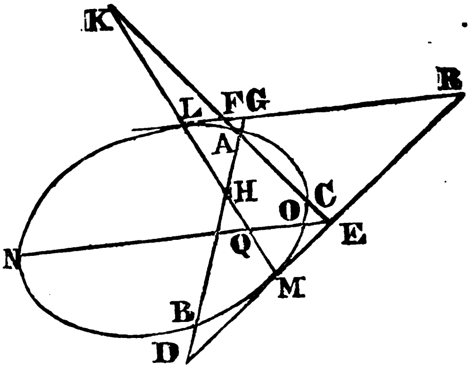


Let CB and DA be drawn, meeting EF in E and F; if CB and DA be parallel, let there be taken in EF, a point G, so that the square of EG may be to the square of GF, as the rectangle AED to the rectangle BFC (*Cor.* 3. 23. 6 and *Cor.* 1. 10. 6 *Eu.*), G is the point of contact (*Cor.* 1. 14. 1 *Sup.*); but if the right lines DA and CB meet each other, as in H, take in EF, a point G, so that the square of EG may be to the square of GF, in a ratio compounded of the ratios of the rectangle AED to the rectangle BFC, and of the rectangle CHB to the rectangle DHA, G is the point of contact (83. 1 *Sup.*)

If the segment EG of the tangent, between the contact, and the secant which is most remote from it, be less than the segment EF of the same, between the secants, the point of contact should be taken on the part of F, which is towards E; but if the segment of the tangent, between the contact, and secant most remote from it, be greater than the segment of the same, between the secants, the point of contact should be taken on the part of the point F, which is remote from E. Describe a conick section passing through the five points A, B, C, D and G (89. 1 *Sup.*), and what was required is done.

PROP. XCI. PROB.

Three points (A, B and C) in a conick section being given, and two right lines (DE and FG) touching it, being given by position, to describe the section.

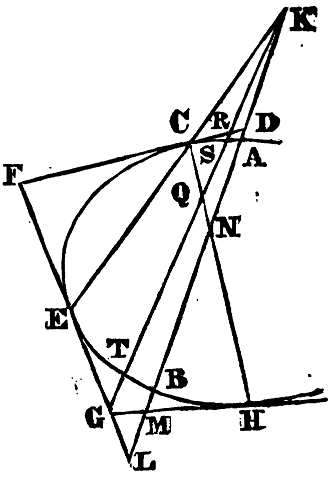


Through two of the given points A and B, draw a right line, meeting the given tangents in D and G; and thro' A and C, a right line, meeting the same tangents in E and F: in GD and FE, take the points H and K, so that the square of DH may be to the square of GH, as the rectangle ADB is to the rectangle AGB (*Cor.* 3 23. 6 and *Cor.* 1. 10 6 *Eu.*), and the square of EK to the square of FK, as the rectangle CEA is to the rectangle AFC (*by the same*); but the points H and K may be taken either between the points D and G, E and F, or without the same; draw KH meeting the tangents in L and M; the points L and M are the points of contact.

For if L and M be supposed to be the contacts, placed somewhere in the tangents, and through any of the four points F, G, D and E, as E, in one of the tangents DE, a right line EN be drawn parallel to the other tangent FG, meeting the section in N and O, and in EN be taken EQ, a mean proportional between EO and EN (13. 6 *Eu.*); the rectangle OEN or (17. 6 *Eu.*), the square of EQ, is to the square of LF, as the rectangle CEA is to the rectangle AFC (*Cor.* 6. 14. 1 *Sup.*), or, which is equal (*Constr.*), as the square of KE is to the square of KF, therefore QE is to LF, as KE is to KF (22. 6 *Eu.*), and alternating, QE to KE, as LF to KF (16. 5 *Eu.*), whence, the angles QEK and LFK being equal (29. 1 *Eu.*), the angles KQE and KLF are equal (6. 6. *Eu.*), therefore, the angles LQE and QLF being together equal to two right angles (29. 1 *Eu.*), the angles KLF and QLF are together equal to two right angles, and so the right lines KL and LQ, and therefore the points K, L and Q are in the same right line, (14. 1 *Eu.*); and for a like reason, the tangents meeting in R, and the rectangle OEN, or (*Constr. and* 17. 6 *Eu.*), the square of EQ being to the square of EM, as the square of RL is to the square of RM (14. 1 *Sup.*), the points L, Q and M are in a right line; since then the point Q is in the right line joining K to either L or M, the points K, L and M are in a right line, and so the point K in the right line joining L and M. In like manner, H being taken in the right line GD, by a similar law, as K in the right line EF, the point H may be shewn to be in the right line joining the points L and M.—Therefore the contacts L and M are in the right line joining the points K and H; which contacts being given, let a section be described through them and the three given points (89. 1 *Sup.*), and what was required is done.

PROP. XCII. PROB.

Two points (A and B) in a conick section being given, and three right lines (CD, EF and GH) touching it, being given by position, to describe the section.



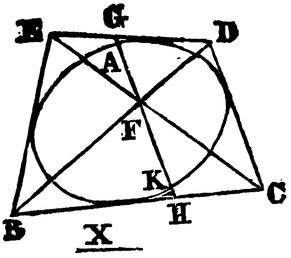
Through the given points A and B, draw a right line, meeting the given tangents in D, M and L, and take therein the point K, so that the square of KL may be to the square of KD, as the rectangle BLA is to the rectangle ADB (*Cor.* 3. 23. 6 and *Cor.* 1. 10. 6 *Eu.*); also the point N so, that the square of DN may be to the square of MN, as the rectangle ADB is to the rectangle BMA (*by the same*).

From K draw KG to the intersection G of the tangents EF and HG, meeting CD in R, and the section in S and T, in KG take the point Q, so that RQ may be to QG as KR is to KG (*Cor.* 1. 10. 6 *Eu.*); draw QN, which produce to meet the section in C and H; draw KC, which produce to meet FG in E; the points C, E and H are the points of contact, through which three points and the two given ones A and B, describe the section (89. 1 *Sup.*), and what was required is done.

For since the square of DN is to the square of MN, as the rectangle ADB is to the rectangle BMA (*Constr.*), the point N is in the right line joining the contacts of the tangents CD and GH (82. 1 *Sup.*), and since the square of KL is to the square of KD, as the rectangle BLA is to the rectangle ADB (*Constr.*), the point K is in the right line joining the contacts of the tangents CD and EL (82. 1 *Sup.*); but, because the square of RQ is to the square of QG, as the square of KR is to the square of KG (*Constr.* and 22. 6 *Eu.*), or which is equal, because of the tangents CD and EL, as the rectangle SRT is to the rectangle TGS (82. 1 *Sup.*), the point Q is in the right line joining the contacts of the tangents CD and GH (*by the same*); but the point N is above proved to be in the right line joining the same contacts; therefore NQ, being drawn and produced as necessary, determines the contacts C and H; and since the point K is above shewn to be in the right line joining the contacts of the tangents CD and EL, and C is shewn to be the contact of the tangent CD, the right line KC being drawn, and produced as necessary to meet the tangent EL, determines the contact E; and so five points A, C, E, B and H in the section are given, as mentioned above.

PROP. XCIII. PROB.

A point (A) in a conick section being given, and four right lines (BC, CD, DE and EB) touching it, being given by position, to describe the section.



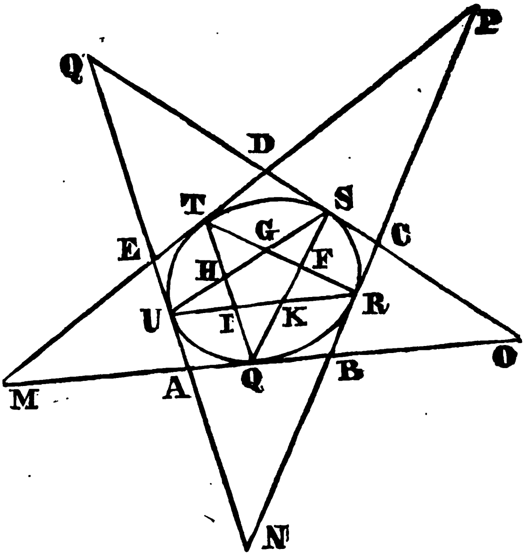
Let BCDE be a quadrangle formed by the four given tangents, draw its diagonals BD and EC intersecting each other in F, join AF, and produce it as necessary, to meet two of the tangents as BC and ED in H and G; take a right lihe X to which AH is in the same ratio, as the square of FH is to the square of FG (*Cor.* 2. 23. 6 *Eu.*), and divide GH in K, so that KH may be to

KG, as AG is to X (*Cor.* 1. 10. 6 *Eu.*) ; the rectangle AHK is to the rectangle AGK in a ratio compounded of the ratios of AH to AG and of HK to GK (23. 6 *Eu.*), or, AG being to X, as HK is to GK (*Constr.*), of the ratios of AH to AG and of AG to X, or (*Def.* 13. 5 *Eu.*), as AH is to X, or which is equal (*Constr.*), as the square of FH is to the square of FG.

Since then the point F is in the right line joining the contacts of the tangents ED and BC (84. 1 *Sup.*), and the rectangle KHA is to the rectangle AGK, as the square of FH is to the square of FG ; the points G and H being in the tangents, and the point A in the section, the point K is also in the section (82. 1 *Sup.*) ; whence, two points A and K in the section being given, and three right lines touching it, being given by position, the section may be described (92. 1 *Sup*).

PROP. XCIV. PROB.

Five right lines (AB, BC, CD, DE and EA) touching a conick section, being given by position, to describe the section.



Let ABCDE, be the quinquelateral figure contained by the tangents, let AB be called the first side, BC the second side, and so on; let BCDM be the quadrangle contained by the four first sides, and let its diagonals meet in F; the first side AB of the quinquelateral figure being now omitted, let CDEN be the quadrangle formed by its other sides, of which quadrangle let the diagonals meet in G; let FG be drawn and produced to meet the second and fourth sides BC and DE in the points T and R.

Proceed thus round the figure, leaving out successively the sides BC, CD and DE, and drawing the diagonals of the quadrangles, intersecting each other in H, I and K; and let GH, HI, IK and KF be drawn, and produced to meet the tangents in the points U and S, T and Q, U and R, Q and S. Through the points Q, R, S, T and U describe a conick section (89. 1 *Sup.*), and what was required is done.

For since both F and G are in the right line joining the contacts of the tangents BC and ED (84. 1 *Sup.*), the intersections T and R of the right line FG with the tangents BC and ED are the points of contact of these tangents; in like manner it may be proved, that the points S, U and Q are the points of contact in the tangents CD, EA and AB; therefore the conick section described through the points Q, R, S, T and U touches the sides of the quinquelateral figure in these points, and of course what was required is done.

SUPPLEMENT
TO THE FIRST SIX BOOKS OF
EUCLID'S ELEMENTS OF GEOMETRY.

BOOK II.

ON SOLIDS.

DEFINITIONS.

1. *A solid,* is that which has length, breadth and thickness.

2. *The bounds of a solid,* are surfaces.

3. A right line is said to be *perpendicular* to a plain, when it is perpendicular to all right lines, which can be drawn in that plain, from the point whereon it insists.

4. One plain is said to be *perpendicular* to another, when all right lines drawn in the one, perpendicular to the line of common section, are perpendicular to the other.

5. The *inclination* of a right line to a plain, is the acute angle, contained under the same right line, and a right line, joining the points, wherein it, and a perpendicular, let fall from any point therein on the plain, meet the same plain.

6. The *inclination* of one plain to another, is the acute angle, contained under two right lines drawn in the same plains, perpendicular to the line of common section, from the same point therein.

7. *Parallel plains,* are such, as being ever so much produced in any direction whatever, do not meet.

8. *A solid angle,* is that, which is made by the meeting of more than two plain angles, which are not in the same plain, in the same point.

9. *A pyramid,* is a solid figure, contained by plains, which are constituted between a plain, and a point without it in which they meet.

10. *A prism*, is a solid figure, contained by plain figures, of which, two which are opposite, are equal, similar and parallel to each other, and the others, parallelograms.

11. *A parallelopiped*, is a solid figure, contained by six quadrilateral figures, whereof every opposite two are parallel.

12. *A sphere*, is a solid bounded by one curve surface, every where equally distant from a point within it.

13. That point is called its *centre*.

14. *A diameter of a sphere*, is a right line passing its centre, and terminated both ways by its surface.

15. *A radius* or *semidiameter of a sphere*, is a right line drawn from the centre to any part of its surface.

16. When a sphere is supposed to be formed by the revolution of a semicircle about its diameter, which remains unmoved, the diameter, about which the semicircle revolves, is called, *the axis of the sphere*.

17. *A cone*, is a solid, bounded by a circle, and a surface, in which, any right line, drawn from the circumference of the circle, to a point without it, wholly lies.

18. That point is called the *vertex* of the cone.

19. The circle is called the *base* of the cone.

20. A right line, joining the vertex to the centre of the base, is called the *axis* of the cone.

21. The curve surface, intercepted between the circumference of the base and the vertex, is called, *the conical surface*.

22. Two conical surfaces, so meeting in a common vertex, that all right lines passing through the common vertex, and coinciding with the surface of one of them, coincide also with the surface of the other, are called, *opposite surfaces*.

23. *A right cone*, is one, whose axis is perpendicular to its base.

24. *An oblique cone*, is one, whose axis is not perpendicular to its base.

25. *A cylinder*, is a solid, bounded by two equal and parallel circles, and a surface in which any right line connecting the circumferences of the circles, and parallel to the right line joining their centres, wholly lies.

26. One of the circles, on which the cylinder is supposed to stand, is called its *base*.

27. The right line, joining the centres of the equal and parallel circles, is called the *axis* of the cylinder.

28. The curve surface, between the circumferences of the circles, is called the *cylindrical surface*.

29. *A right cylinder,* is one whose axis is perpendicular to the base.

30. *An oblique cylinder,* is one, whose axis is not perpendicular to the base.

31. A *tetrahedron,* is a solid figure, contained by four equal and equilateral triangles.

32. A *cube* or *hexahedron,* is a solid figure contained by six equal squares.

33. An *octohedron,* is a solid figure, contained by eight equal and equilateral triangles.

34. A *dodecahedron,* is a solid figure, contained by twelve regular pentagons.

35. An *icosihedron,* is a solid figure, contained by twenty equal and equilateral triangles.

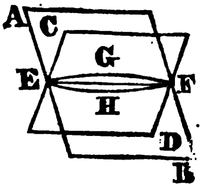
POSTULATE.

That; by any right line, and any point without it, a plain may pass, and be produced at pleasure.

Cor. Hence a plain may pass by any three points, or by any two right lines meeting each other, and therefore, by all the angles and sides of any rectilineal triangle.

PROP. I. THEOR.

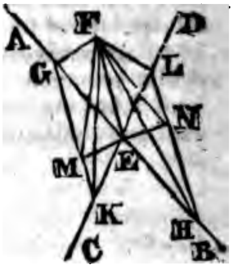
The common section (EF) of two plains (AB and CD), is a right line.



For if EF be not a right line, drawing the right lines EGF and EHF on the plains AB and CD, these right lines would contain a space, which is absurd (*Ax.* 10. 1 *Eu*).—Therefore the common section of the plains AB and CD is a right line.

PROP. II. THEOR.

A right line (EF), which is perpendicular to two right lines (AB and CD) meeting each other at the common section (E), is perpendicular to the plain passing by the same right lines.

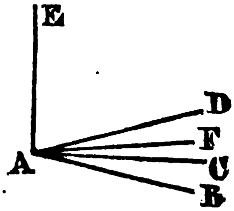


On EA and EB take EG and EH equal to each other, and on EC and ED take EK and EL equal to each other; draw GK and LH, and through E, in the plain passing by AB and CD, draw any right line MN meeting GK and LH in M and N; and let FG, FM, FK, FN and FH be drawn.

Because EG is equal to EH (*Constr.*), EK to EL (*by the same*), and the angle KEG to LEH (15. 1 *Eu.*), the angles EGK and EHL, and the right lines GK and HL are equal (4. 1 *Eu.*), and since in the right angled triangles FEG and FEH, the right lines GE and EH are equal, and FE common, FG and FH are equal (4. 1 *Eu.*); in like manner FK and FL may be proved equal; therefore, the triangles GFK and HFL being mutually equilateral, the angles FGK and FHL are equal (8. 1 *Eu.*); and since EG is equal to EH (*Constr.*), the angle GEM to HEN (15. 1 *Eu.*), and the angles EGM and EHN have been proved equal, EM and EN, as also GM and HN are equal (26. 1 *Eu.*), whence, FG and FH having been above proved equal, as also the angles FGM and FHN, the right line FM is equal to FN (4. 1 *Eu.*); therefore, EM having been already shewn to be equal to EN, and EF being common to the two triangles EMF and ENF, the angles FEM and FEN are equal (8. 1 *Eu.*), and therefore EF is perpendicular to MN (*Def.* 20. 1 *Eu*). In like manner EF might be shewn to be perpendicular to any right line drawn through E in the plain passing by AB and CD, it is therefore perpendicular to the same plain (*Def.* 3. 2. *Sup*).

PROP. III. THEOR.

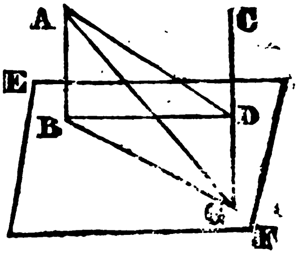
Three right lines (AB, AC and AD), which are perpendicular to the same right line (EA), at the same point (A), are in the same plain.



For if one of them, as AD, be not in the plain passing by the two others AB and AC, let some other right line AF be the common section of the plains passing by AB and AC, and by AE and AD; and since AE is perpendicular to to AB and AC (*Hyp.*), it is perpendicular to the plain passing by them (2. 2. *Sup.*); therefore EAF is a right angle (*Def.* 2. 2 *Sup.*), and therefore equal to EAD, which is also a right angle (*Hyp.*), whole and part, which is absurd (*Ax.* 9. 1 *Eu.*); therefore the right lines AB, AC and AD are in the same plain.

PROP. IV. THEOR.

Two right lines (AB and CD), which are perpendicular to the same plain (EF), are parallel to each other.



Draw in the plain EF, the right line BD, and in the same plain, draw DG perpendicular to BD, and equal to AB, and let BG, AG and AD be joined.

Because in the triangles BAD and BGD, the sides AB and DG are equal (*Constr.*), BD common, and the angles ABD and BDG equal, being both of them right angles (*Hyp. Def.* 3. 2. *Sup. and Constr.*), AD and BG are equal (4. 1 *Eu.*), whence, the triangles AGB and AGD having also AB equal to DG, and AG common, the angle ADG is equal to the angle ABG (8. 1 *Eu.*), and therefore a right one; whence the right lines DC, DA and DB, being each of them perpendicular to DG, are in the same plain (3. 2 *Sup.*), in which plain is also AB (*Cor. Post B.* 2 *Sup.*); since then AB and CD are in the same plain, and the angles ABD and CDB right angles (*Hyp. and Def.* 3, 2. *Sup.*), AB and CD are parallel to each other (28. 1 *Eu*),

PROP. V. THEOR.

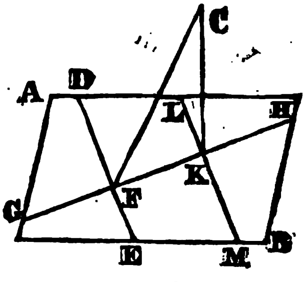
If one (AB, see fig. to prec. prop.), of two parallel right lines (AB and CD), be perpendicular to a plain (EF), the other (CD) is perpendicular to the same plain.

Draw in the plain EF, the right line BD, and in the same plain, DG perpendicular to BD, and equal to AB, and let BG, AG and AD be joined.

Because in the triangles BAD and BGD, the sides AB and DG are equal (*Constr.*), BD common, and the angles ABD and BDG equal, being each of them right, AD and BG are equal (4. 1 *Eu.*), whence the triangles AGB and AGD having also AB and DG equal, and AG common, the angle ADG is equal to ABG (8. 1 *Eu.*), and therefore a right one; therefore DG being perpendicular to BD and AD, is perpendicular to the plain passing by them (2. 2. *Sup.*), in which plain is DC, since AB and DC are in the same plain (*Hyp. and Def.* 34. 1 *Eu.*), therefore DG is perpendicular to DC (*Def.* 3. 2 *Sup.*); but AB and CD being parallels (*Hyp.*), and the angle ABD right (*Hyp.* and *Def.* 3. 2 *Sup.*), the angle CDB is also right (29. 1 *Eu.*); therefore CD being perpendicular to both BD and DG, is perpendicular to the plain EF in which they are (2. 2 *Sup*).

PROP. VI. PROB.

On a given plain (AB), from a given point (C) not therein, to let fall a perpendicular.



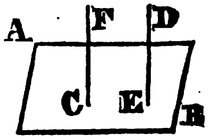
Having drawn DE at pleasure in the plain AB, let fall thereon from the point C, the perpendicular CF (12. 1 *Eu.*), in the plain AB by F draw GH perpendicular to DE (11. 1 *Eu.*), and let fall thereon from the point C the perpendicular CK, (12. 1 *Eu.*), which is the perpendicular required.

For through K having drawn LM parallel to DE (31. 1 *Eu.*), because DF is perpendicular to FC and FK (*Constr.*), it is perpendicular to the plain passing by them (2. 2 *Sup.*), in which plain is CK (*Cor. Post. B.* 2. 2. *Sup.*); whence LK, being parallel to DF (*Constr.*), is perpendicular to the same plain (5. 2 *Sup.*), and therefore the angle CKL is a right angle (*Def.* 3. 2 *Sup.*); whence, the angle CKH being also a right angle *Constr.*), CK is perpendicular to the plain passing by KL and KH (2. 2 *Sup.*), or to the plain AB.

PROP. VII. PROB.

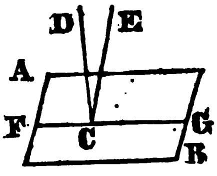
To a given plain (AB), at a given point therein (C), to erect a perpendicular.

From any point D without the plain AB, let DE be drawn perpendicular to it (6. 2 *Sup.*), and through C draw CF parallel to ED (31. 1 *Eu.*); and because ED is perpendicular to the plain AB (*Constr.*), CF, which is parallel to it, is perpendicular to the same plain (5. 2 *Sup.*), and therefore what was required is done.



PROP. VIII. THEOR.

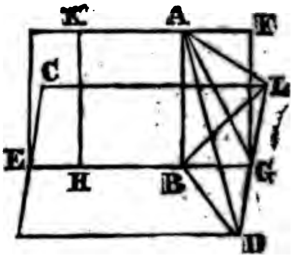
From the same point in a plain, there cannot be two right lines at right angles to the plain, on the same side of it: and there can be but one perpendicular to a plain from a point above it.



For, if it be possible, let two right lines CD and CE be at right angles the same plain AB, from the same point C in the plain, and on the same side of it; and let a plain pass by CD and CE (*Cor. Post. B.* 2 *Sup.*), the common section of which with the plain AB is a right line passing by C (1. 2 *Sup.*); let FCG be their common section, therefore the right lines CD, CE and FCG are in the same plain; and because CD is perpendicular to the plain AB (*Hyp.*), it is perpendicular to every right line drawn through C in that plain (*Def.* 3. 2 *Sup.*), and therefore to the right line FCG, therefore the angle DCG is a right angle; for a like reason ECG is a right angle; therefore the angles ECG and DCG are equal, part and whole, which is absurd. And from a point above a plain, there can be but one perpendicular to the plain, for if there could be two, they would be parallel to each other (5. 2 *Sup.*), which is absurd.

PROP. IX. THEOR.

If a right line (AB) be perpendicular to a plain (CD), all the plains drawn thereby are perpendicular to the same plain.



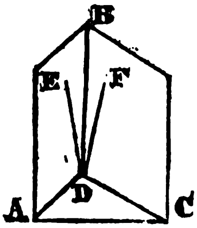
Let EF be a plain passing by AB, and EG the common section thereof with the plain CD, from any point wherein H in the plain EF, let HK be drawn perpendicular to EG (11. 1 *Eu.*), which, the angles ABH and KHB being each of them right angles, is parallel to AB (28. 1 *Eu.*), and therefore perpendicular to the plain CD (*Hyp.* & 5. 2 *Sup.*); the like might be proved of any right line drawn in the plain EF perpendicular to the common section EG, therefore the plain EF perpendicular to the plain CD (*Def.* 4. 2 *Sup.*); the like might be proved of any other plain passing by AB.

Cor. If from the point (B), wherein a perpendicular (AB) to a plain (CD) meets it, a perpendicular (BG) be drawn to any right line (DL) in the same plain; a right line (AG), joining any point (A) in the perpendicular to the plain, to the point (G) where the perpendicular to the right line meets it, is perpendicular to the same right line (DL).

In the right line DL take equals GD and GL, and join BD, BL, AD and AL; the triangles BGD and BGL right angled at G, having BG and GD severally equal to BG and GL, the right lines BD and BL are equal (4. 1. *Eu.*); whence, the triangles ABD and ABL having AB common, and the angles at B right (*Hyp. and Def.* 3. 2 *Sup.*), AD and AL are equal (4. 1 *Eu.*); whence, the triangles AGD and AGL having AG common, and GD equal to GL, the angles AGD and AGL are equal (8. 1 *Eu.*), and therefore right (*Def.* 20. 1 *Eu*).

PROP. X. THEOR.

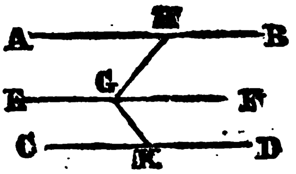
If two plains (AB and CB), intersecting each other, be perpendicular to a third (ADC); their common section (DB) is perpendicular to the same plain.



If DB be not perpendicular to the plain ADC, draw from the point D in the plain AB, the right line DE perpendicular to AD (11. 1 *Eu.*); and in the plain CB, from the point D, draw DF perpendicular to CD (*by the same*); and because the plain AB is perpendicular to the plain ADC, and DE is drawn in the plain AB, perpendicular to AD their common section, DE is perpendicular to the plain ADC (*Def.* 4. 2 *Sup*). In like manner DF may be proved to be perpendicular to the plain ADC. Therefore two right lines DE and DF are at right angles to the same plain ADC, on the same side of it, at the same point D, which is absurd (8. 2. *Sup.*). Therefore there cannot be any right line perpendicular to the plain ADC at the point D, except DB; therefore DB is perpendicular to the plain ADC.

PROP XI. THEOR.

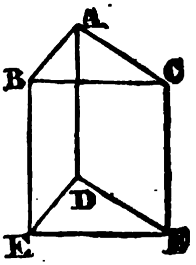
Two right lines (AB and CD), parallel to the same right line (EF), which is not in the same plain with them, are parallel to each other.



From any point G in EF, let two perpendiculars GH and GK to EF be drawn (11. 1 *Eu.*), one in the plain passing by EF and AB, and the other in that passing by EF and CD, meeting AB and CD in H and K; and EF being perpendicular to GH and GK (*Constr.*), is perpendicular to the plain passing by them (2. 2 *Sup.*); whence AB and CD being parallel to EF, are perpendicular to the same plain (5. 2 *Sup.*), and therefore parallel to each other (4. 2. *Sup*).

PROP. XII. THEOR.

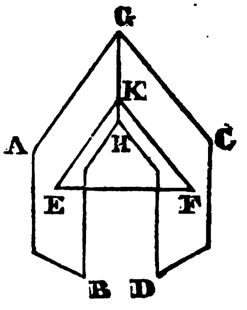
If two right lines (AB and AC), meeting each other (as in A), be parallel to two others (DE and DF), likewise meeting each other (as in D), though not in the same plain with them; the first two and the last two contain equal angles.



Let AB and DE be taken equal to each other, and also AC and DF, and let BC, EF, BE, AD and CF be drawn. And since AB and DE are equal and parallel, BE and AD are equal and parallel (33. 1 *Eu.*); in like manner CF may be shewn to be equal and parallel to AD; therefore BE and CF are equal and parallel to each other (*Ax.* 1. 1 and 30. 1 *Eu.*), and therefore BC and EF are equal (33. 1 *Eu.*); whence the triangles ABC and DEF being mutually equilateral, the angles BAC and EDF are equal (8. 1 *Eu*).

PROP. XIII. THEOR.

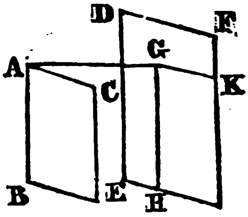
Plains (AB and CD), to which the same right line (EF) is perpendicular, are parallel.



For if not, let them, being produced, meet, and let their common section be the right line GH, to any point wherein as K, having drawn EK and FK, the angles KEF and KFE are each of them right angles (*Hyp.* and *Def.* 3. 2 *Sup.*), and so two angles of the triangle KEF are equal to two right angles, which is absurd (17. 1 *Eu.*); therefore the plains AB and CD being produced, do not meet each other, and are therefore parallel (*Def.* 7. 2 *Sup*).

PROP. XIV. THEOR.

If two right lines (AB and AC), meeting each other, be parallel to two others (DE and DF), which meet each other, but are not in the same plain with the former two; the plain (BC), passing by the former two, is parallel to that (EF), passing by the others.

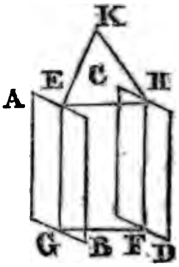


From A, let fall the perpendicular AG on the plain EF (6. 2 *Sup.*), and from the point G, wherein it meets the same, draw GH parallel to DE and GK to DF (31. 1 *Eu.*); and since AB and GH are each of them parallel to DE, they are parallel to each other (11. 2 *Sup.*), and therefore the angle AGH being a right angle (*Constr. and Def.* 3. 2 *Sup.*), GAB is a right angle (29. 1 *Eu.*); in like manner might GAC be shewn to be a right angle; therefore GA being perpendicular to AB and AC, is perpendicular to the plain BC passing by them (2. 2 *Sup.*); whence, the right line AG being likewise perpendicular to the plain EF (*Constr.*), the plains BC and EF are parallel (13. 2 *Sup*).

Cor. Hence it appears how a plain may be found, passing through any given point, as D, parallel to a given plain BC; namely, by drawing in the given plain two right lines AB and AC, from any point therein A, and drawing from the given point D, two right lines DE and DF parallel to AB and AC (31. 1 *Eu*). The plain passing by the right lines DE and DF (*Post. to B.* 2 *Sup.*), is parallel to the given plain BC (14. 2 *Sup.*), as was required to be found.

PROP. XV. THEOR.

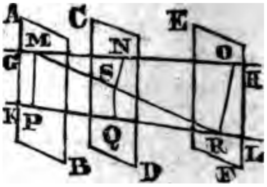
If two parallel plains (AB and CD) be cut by a third (EF); their common sections (EG and HF) are parallel.



For if not, let them, being produced, meet, as in K, and since the plains AB and CD, being produced, coincide with these right lines GE and FH (*Def.* 4 *and* 7. 1 *Eu.*), these plains being produced meet also in K, which is absurd (*Hyp. and Def.* 7. 2 *Sup.*); therefore the common sections GE and FH being produced towards E and H do not meet; in like manner it may be shewn, that they do not meet towards G and F, therefore they do not meet being produced either way, and are therefore parallel (*Def.* 34. 1 *Eu*).

PROP. XVI. THEOR.

Parallel plains (as AB, CD and EF), cut right lines (as GH and KL) proportionally.

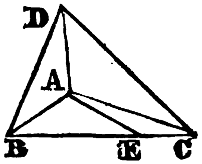


Let the right line GH meet the parallel plains in M, N and O, and KL the same plains in P, Q and R; MN is to NO, as PQ is to QR.

Join MP and OR, also MR meeting the plain CD in S, and join SN and SQ; and because the parallel plains CD and EF are cut by the plain MRO, the common sections NS and OR are parallel (15. 2 *Sup.*); in like manner, since the parallel plains AB and CD are cut by the plain MPR, the common sections MP and SQ are parallel (*by the same*): whence, in the triangle MOR, the right line NS being parallel to OR, MN is to NO, as MS to SR (2. 6 *Eu.*); in like manner, in the triangle MPR, SQ being parallel to MP, MS is to SR, as PQ to QR (*by the same*); whence, the ratios of MN to NO, and of PQ to QR, being each equal to that of MS to SR, are equal to each other (11. 5 *Eu*).

PROP. XVII. THEOR.

Of three plain angles forming a solid angle, any two whatever are greater than the third.

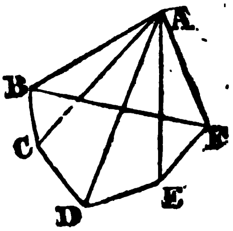


Let the solid angle A be formed by three plain angles BAC, CAD and DAB. Any two of them are greater than the third.

If the angles BAC, CAD and DAB be equal, it is evident that any two of them are greater than the third; if not, let BAC be that angle which is not less than either of the other two, and is greater than one of them BAD, and from the angle BAC take BAE equal to BAD (23. 1 *Eu.*), take AD and AE equal to each other, through E draw the right line BEC meeting AB and AC in B and C, and draw DB and DC; and because in the triangles BAD and BAE, the side AD is equal to AE (*Constr.*), AB common, and the included angles BAD and BAE equal (*Constr.*), BD is equal to BE (4. 1 *Eu.*); but BD and DC together are greater than BC (20. 1 *Eu.*), taking from each the equals BD and BE, there remains DC greater than EC (*Ax.* 5. 1 *Eu.*); whence, AD and AC being severally equal to AE and AC, the angle DAC is greater than the angle EAC (25. 1 *Eu.*), to which adding the equal angles BAD and BEA, the angles BAD and DAC together, are greater than BAE and EAC together, or than the whole angle BAC (*Ax.* 4. 1 *Eu.*); and the angle BAC, being not less than either of the other two, is with either of them, greater than the other.

PROP. XVIII. THEOR.

The plain angles, which constitute any solid angle, are together less than four right angles.



Let A be a solid angle, contained by any number of plain angles BAC, CAD, DAE, EAF and FAB, these are together less than four right angles.

Let the plains which contain the solid angle A be cut by another plain BCDEF; and of the three plain angles which contain the solid angle at B, the angles ABF and ABC are together greater than the third CBF (17. 2 *Sup.*); for the same reason, of the three plain angles which contain each of the solid angles at C, D, E and F, the two which are at the bases of the triangles having their common vertex at A, are together greater than the third, which is one of the angles of the figure BCDEF; therefore all the angles at the bases of the triangles having their common vertex at A are together greater than all the angles of the figure BCDEF; but all the angles of the figure BCDEF are equal to twice as many right angles, except four, as the figure has sides (*Cor.* 1. 32. 1 *Eu.*), therefore all the angles at the bases of the triangles having their common vertex at A are greater than twice as many right angles, except four, as the figure has sides; and all the angles of these triangles are equal to twice as many right angles as the figure has sides (32. 1 *Eu.*), therefore the angles of these triangles which are at their common vertex A, being those which contain the solid angle A, are less than four right angles:

Cor. From this proposition it follows, that there can be no more than five solids contained by equilateral and equiangular plain figures, or as they are usually called, *regular solids*, namely, three contained by equilateral triangles, one by squares, and one by regular pentagons.

For a solid angle cannot be contained by two plain angles, three at least are required.

And since the three angles of an equilateral triangle are equal to two right angles (32. 1 *Eu.*), six such angles are equal to four right angles, and therefore cannot constitute a solid angle (*by this prop.*); and since six angles of an equilateral triangle are equal to four right angles, three, four, or five such angles are less than four right angles, and can therefore constitute a solid angle, as is manifest from this proposition; but three such angles form the angle of a tetrahedron or equilateral pyramid, see Def. 31. 2 Sup.; four such angles form the angle of an octohedron, see Def. 33. 2 Sup.; and five such angles form the angle of an icosihedron, see Def. 35. 2 Sup.

Three angles of a square form the angle of a cube or hexahedron, see Def. 32. 2 Sup.; four such angles are equal to four right angles, and therefore cannot constitute a solid angle.

And since the five angles of a regular pentagon are equal to six right angles (*Cor.* 1. 32. 1 *Eu.*), any one of its angles is equal to a right angle and a fifth of a right angle, and therefore three such angles are equal to three right angles and three fifths, and three such angles form the angle of a dodecahedron, see Def. 34. 2. Sup.; but four such angles are equal to four right angles and four fifths, and therefore cannot form a solid angle (*by this prop*).

And since six angles of an equilateral triangle, or four angles of a square, are equal to four right angles, and four angles of a regular pentagon are greater than four right angles, therefore more than six angles of an equilateral triangle, or than four of a square, or than four of a regular pentagon, are greater than four right angles, and therefore cannot constitute a solid angle (*by this prop*).

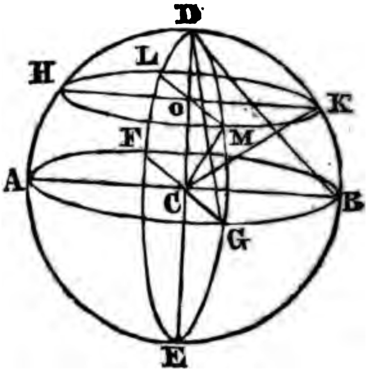
And since the six angles of a regular hexagon are equal to eight right angles (*Cor.* 1. 32. 1 *Eu.*), three such angles are equal to four right angles, and therefore cannot constitute a solid angle (*by this prop*); neither therefore can any greater number.

And since three angles of a regular hexagon are equal to four right angles, three angles of a regular heptagon, or of any regular polygon of more than six sides, are greater than four angles, as also easily follows from Cor. 1. 32. 1 Eu., therefore all regular polygons of more than five sides are incapable of forming a solid angle, and therefore there can be no more regular solids than the five mentioned in this corollary.

Schol.—It is manifestly supposed in this proposition, that when the solid angle is contained by more than three plain angles, any of the legs of the plain angles which form the solid angle, as AC, falls without the plain passing by the two adjacent legs AB and AD.

PROP. XIX. THEOR.

The section of a plain with the surface of a sphere (ADBE), is a circle, whose centre is in the diameter of the sphere which is perpendicular to the plain.



Part 1. If the section pass through the centre C of the sphere, as the section AFBG, the proposition is manifest, since all right lines, as CB and CG, drawn from the centre C to the surface of the sphere, and therefore to the perimeter of the section, are equal (*Def.* 12. 2 *Sup.*), which section is therefore a circle (*Def.* 10. 1 *Eu*).

Part 2. But if the section do not pass through the centre of the sphere, as the section HLKM, let CO be a perpendicular drawn from the centre C to the plain HLKM (6. 2 *Sup.*); through O draw, in the plain HLK, any two right lines whatever HK and LM, meeting the section HLKM in the points H and K, L and M, and draw CM and CK; and because the triangles COK and COM have the angles at O right (*Constr.* and *Def.* 3. 2 *Sup.*), the sides CK and CM equal to each other (*Def.* 12. 2 *Sup.*), and CO common to both, the sides OK and OM are equal (*Cor.* 7. 6 *Eu.*); in like manner, may all right lines drawn from O to the perimeter of the section HLKM be proved equal to each other, which section is therefore a circle (*Def.* 10. 1 *Eu.*), whose centre is in the diameter of the sphere DE, which is perpendicular to the plain HLKM.

Cor. From this proposition it appears, how the centre of a given sphere may be found, namely, by finding the centre O of any circle HLK formed by the intersection of a plain with its surface (1. 3 *Eu.*), drawing through that centre a perpendicular to the plain (7. 2 *Sup.*), to meet the surface of the sphere both ways as in D and E, and bisecting DE in C; the point C is the centre of the sphere.

If the centre of the sphere be in DE, 'tis plain, C must be that centre; if the centre were in any point without DE, right lines being drawn, in the plain passing by DE and that point from that point to O and the intersections of the same plain with the circle HLK, a like absurdity might be shewn to follow, as in 1. 3 *Eu.*

PROP. XX. THEOR.

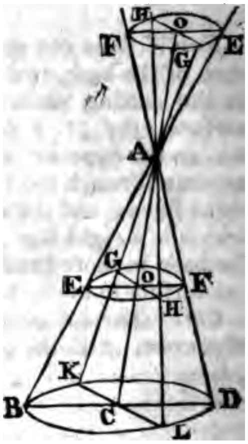
Any plain, passing through the vertex of a cone, and cutting the circumference of its base, cuts the opposite surfaces in two right lines, and in them only.

For if from the points in which this plain cuts the circumference of the base, two right lines be drawn to the vertex, they are in the cutting plain (*Def.* 4 and 7. 1 *Eu.*), and in the conical surface (*Def* 17. 2 *Sup.*), and, being produced beyond the vertex, in the opposite surface (*Def.* 22. 2 *Sup.*); therefore the plain passing through the vertex cuts the opposite surfaces in these two right lines; and since the intersection of this plain with the base, which is a right line (1. 2 *Sup.*), cannot cut the circumference of the base in more than two points, that plain cannot cut the conical surface in more than these two right lines.

Cor. The intersection of a plain, passing through the vertex of a cone, with the conical surface and base, is a triangle.

PROP. XXI. THEOR.

If either of the opposite surfaces of a cone be cut by a plain parallel to the base, the section is the circumference of a circle whose centre is in the axis of the cone.



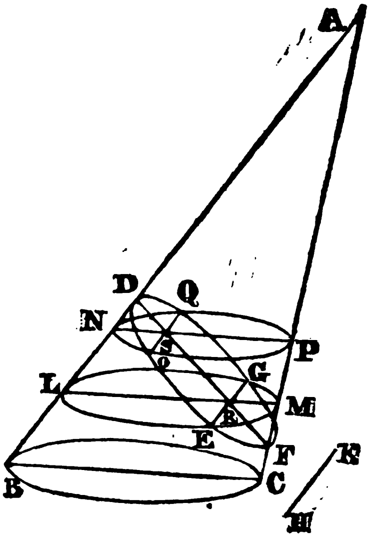
Let ABD be a cone, whose vertex is A, and its base the circle BKDL, of which C is the centre; AC is its axis (*Def.* 20. 2 *Sup.*); let EGFH be the section of a plain parallel to the base, with one of the opposite surfaces; EGFH is a circle, whose centre is in the axis AOC.

Let the cone be cut by any two plains passing by the axis AC, making the triangles ABD and AKL (*Cor.* 20. 2 *Sup.*), and let these triangles, produced if necessary, meet the plain EGFH in the right lines EOF and GOH; because of the parallel plains, the right lines EF and BD, as also GH and KL, are parallel (15. 2 *Sup.*); therefore the triangles AOF and ACD, as also AOH and ACL, are equiangular, and therefore the ratios of CD to OF, and of CL to OH, are each of them equal to that of AC to AO (4. 6 and 16. 5 *Eu.*), and therefore to each other (11. 5 *Eu.*); whence, CD and CL being, on account of the circular base of the cone, equal, OF and OH are equal (14. 5 *Eu*). In like manner it may be shewn, that any other two right lines, drawn from O to the section EGFH are equal; since therefore all right lines drawn from O to that section are equal, that section is the circumference of a circle whose centre is O, and therefore in the axis AC of the cone.

Cor. Hence it follows, that any diameter of such a section passes through the axis of the cone; and any right line, drawn in the plain of such a section, through the axis of the cone, is a diameter of the section.

PROP. XXII. THEOR.

If an oblique cone be cut by a plain passing by the axis and at right angles to the base, and be cut by another plain perpendicular to the former, and cutting off from the triangle, which is the section of the former with the cone, a triangle similar to that triangle, but placed subcontrarily, namely, so that the equal angles be at different right lines formed by the section of the first plain with the conical surface; the section of the conical surface by the latter plain, is the circumference of a circle, whereof the intersection of the two drawn plains is a diameter.



Let ABC be a cone, A its vertex, BC its base, and the triangle ABC the section with the cone, of a plain passing by the axis and at right angles to the base (*Cor.* 20. 2 *Sup.*); let DEFG be the section with the cone, of a plain perpendicular to the plain ABC, and cutting off from the triangle ABC a triangle AFD similar to ABC, but placed subcontrarily, namely, so that the angle AFD be equal to the angle ABC; the section DEFG is a circle, of which the intersection DF of the plains ABC and DEF is a diameter.

Let the right line HK be the intersection of the plain DEF with the plain of the base; from any point E in the section DEFG, draw ERG parallel to HK, meeting DF in R, and through R, in the plain ABC, draw LM parallel to BC; the plain passing by ERG and LRM is parallel to the base BC (14. 2 *Sup.*), and therefore its intersection with the surface of the cone is a circle whose diameter is LM (21. 2 and *Cor.* 21. 2 *Sup.*): and because both the base and the plain DEF are perpendicular to the plain ABC (*Hyp.*), their intersection HK is perpendicular to the same plain produced as necessary (10. 2 *Sup.*), and therefore ER, which is parallel to HK (*Constr.*), is perpendicular to the plain ABC (5. 2 *Sup*), and therefore to both the right lines LM and DF (*Def.* 3. 2 *Sup.*); whence, because of the circle LEM. the square of EK is equal to the rectangle LRM (3 *and* 35. 3 *Eu.*); but the angle MFR is equal to the angle ABC (*Hyp.*), or its equal (29, 1 *Eu.*), DLR, and the angles FRM and DRL are equal (15, 1 *Eu.*), therefore the triangles FRM and DRL are equiangular (32. 1 *Eu.*), and FR is to RM, as LR to RD (4. 6 *Eu.*), and therefore the rectangle DRF is equal to the rectangle LRM (16. 6 *Eu.*), or to the square of ER. Therefore the point E is in the circumference of a circle described about the diameter DF in the plain DEF, for if RE met the circumference of a circle so described, in any other point on the part E of DF, the rectangle DRF would not be equal to the segment of RE between DF and the circle, contrary to 3 and 35. 3 Eu. In like manner any other point in the section DEFG may be proved to be in the circumference of a circle described about the diameter DF in the plain DEF, which section is therefore the circumference of a circle, whereof DF is a diameter.

Scholium. A section of this kind is called, a *subcontrary section*

Cor. From this proposition, the preceding, and Def. 17. 2 Sup., it follows, that any circle formed by a plain parallel to the base of a cone, or by a subcontrary section, may be considered as the base of a cone having the same vertex, as the original one.

PROP. XXIII. THEOR.

If the intersection of a plain not parallel to the base of a cone, with the conical surface, be the circumference of a circle, the section is a subcontrary one.

Let the intersection DEFG, see fig. to the preceding proposition, of a plain, not parallel to the base BC of a cone, with the conical surface, be the circumference of a circle; the section DEFG is a subcontrary one.

Let the conical surface be cut by two plains parallel to the base, making the circles LEMG and NOPQ, and intersecting the plain DEFG in the right lines EG and OQ. which are parallel (15. 2 *Sup*); draw diameters LM and NP, of the circles LEM and NOP, perpendicular to the right lines EG and OQ, and meeting them, in the points R and S; the right lines EG and OQ are bisected in R and S (3. 3 *Eu.*), and LM and NP are parallel, for if any other right line drawn through S in the plain NOP, except NP, were parallel to LM, the angle made by that parallel with SQ would not be equal to the angle LRG, contrary to 12. 2 Sup.; and the diameters LM and NP pass by the axis of the cone (*Cor.* 21. 2 *Sup.*); therefore the plain ANLBCMP passes by the axis of the cone; let the right line DF be the intersection of this plain with the plain of the circle DEFG; since therefore DF passes through the points R and S, and therefore bisects the parallels EG and OQ, it is a diameter of the circle DEFG, for if any other right line, bisecting one of them not passing through the centre, were a diameter, it would bisect it perpendicularly (3. 3 *Eu.*), and would therefore, because of the parallels, be perpendicular to the other (29. 1 *Eu.*), and therefore would bisect that other (3. 3 *Eu.*), which would of course be bisected in two points, which is absurd; and that DF is a diameter appears also from Cor. 2. 32. 1 Sup., and it is perpendicular to the right lines EG and OQ (3. 3 *Eu.*); since therefore EG is perpendicular to the right lines LRM and DRF, it is perpendicular to the plain ABC passing by them (2. 2 *Sup.*), and therefore the plains DEFG and LEMG and of course the plain of the base, are perpendicular to the plain ABC passing by the axis (9. 2 *Sup.*); but because of the circles LEM and DEF, each of the rectangles LRM and DRF are equal to the square of ER (3 and 35. 3 *Eu.*), and therefore to each other, therefore DR is to LR, as RM is to RF (16. 6 *Eu.*), and the angles DRL and FRM are equal (15. 1 *Eu.*), therefore the triangles DRL and MRF are equiangular (6. 6 *Eu.*), and the angle MFR is equal to DLR, or its equal (29. 1 *Eu.*), ABC; therefore the section DEFG is a subcontrary section (*Schol.* 22. 2 *Sup*).

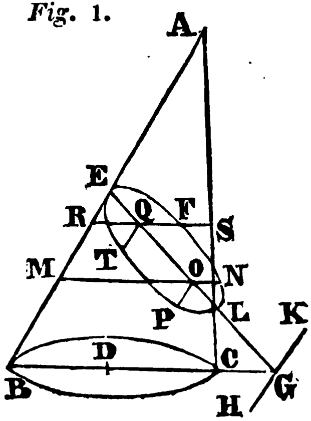
Cor. Hence the section made by a plain with a conical surface, which is neither a subcontrary section, nor made by a plain, which is parallel to the base, is not the circumference of a circle.

PROP. XXIV. THEOR.

If a cone be cut by a plain neither passing through its vertex, nor parallel to its base, nor placed subcontrarily; the section is an ellipse, hyperbola or parabola; the intersection of that plain with a plain passing by the axis of the cone, and a right line drawn through the centre of its base, perpendicular to the common section of that plain with the base of the cone, being a diameter of the section; the section being an ellipse; when this diameter meets the conical surface of the cone twice; a hyperbola, when it meets the opposite conical surfaces; and a parabola, when being parallel to a right line drawn from the vertex of the cone to the circumference of its base, it meets only one of the conical surfaces, and but once.

Let a cone ABC, see fig. 1, 2 and 3, be cut by a plain PEF, neither passing through its vertex, nor parallel to its base, nor placed subcontrarily; let HK be the common section of this plain with the plain of the base, D the centre of the base, DG a right line, drawn through D perpendicular to HK, ABC the triangular section of the cone with the plain passing by the axis and DG, and EG the intersection of the plains PEF and ABC; the section PEF is an ellipse, hyperbola or parabola, of which EG is a diameter; the section being an ellipse, when the diameter EG meets the conical surface of the cone ABC twice, as in E and L in fig. 1; a hyperbola, when it meets the opposite conical surfaces, as in E and L in fig. 2; and a parabola, when the diameter EG, being parallel to a right line AC, drawn from the vertex A of the cone, to the circumference of its base, meets only one of the conical surfaces, as in E in fig. 3, and but once.

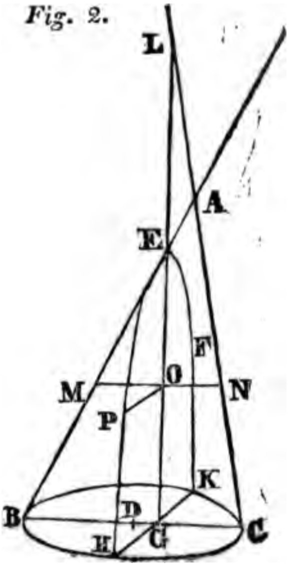
Fig. 1.



Let *first* EG meet the conical surface of the cone ABC in E and L, and let there be taken in the section any point P, and draw PO parallel to HK meeting EL in O, and thro' O draw MN parallel to BC; the plain passing by MN and PO is parallel to that passing by BC and HK (14. 2 *Sup.*); therefore the section by the plain POM is a circle whose diameter is MN (21. 2 and *Cor.* 21. 2 *Sup.*), and because the angle BCH is a right angle (*Hyp.*), MOP is a right angle (12. 2 *Sup.*), therefore the square of PO is equal to the rectangle MON (3 and 35. 3 *Eu*). In like manner, if any other point T be taken in the section PEL, and TQ be drawn to EL parallel to HK or PO, and through Q, RS be drawn parallel to BC, it may be shewn, that the square of TQ is equal to the rectangle RQS; therefore the square of PO is to the square of TQ, as the rectangle EOL is to the rectangle EQL (*Cor.* 1. 7. 5 *Eu.*); but, because of the equiangular triangles EMO and ERQ, LON and LQS, MO is to RQ, as EO is to EQ, and ON is to QS, as OL to QL (4. 6 and 16. 5 *Eu.*), therefore the ratios of the rectangle MON to the rectangle RQS, and of the rectangle EOL to the rectangle EQL, being compounded of these equal ratios (23. 6 *Eu.*), are equal (22. 5 *Eu.*); therefore the square of PO is to the square of TQ, as the rectangle EOL is to the rectangle EQL (11. 5 *Eu*). Let an ellipse be described with the diameter EL, its vertices being E and L, and the ordinate PO (85. 1 *Sup.*), and because the square of TQ is to the square of PO, as the rectangle EQL is the rectangle EOL, the point T is in the ellipse (*Cor.* 3. 40. 1 *Sup.*): in like manner it may be shewn that any other point in the section PEL is in the ellipse; and since the section, being neither a subcontrary one, nor parallel to the base of the cone (*Hyp.*), is not a circle (*Cor.* 23. 2 *Sup.*), it is an ellipse, of which EL is a diameter.

Secondly. Let EG produced meet the opposite conical surface in L, let H and K be the points in which HK meets the circumference of the base of the cone, from any point P in the section HEK, draw PO parallel to HK, meeting EG in O, and through O draw MN parallel to BC; as before, the plain by PO and MO is parallel to that by HG and BC, or to the base of the cone, and the angle POM equal to HGB, and therefore a right angle, therefore the square of PO is equal to the rectangle MON (3 and 35. 3 *Eu.*); in like manner it may be proved, that the square of HG is equal to the rectangle BGC; therefore the square of PO is to the square of HG, as the rectangle MON is to the rectangle BGC (*Cor.* 1. 7. 5 *Eu.*); but, because of the equiangular triangles

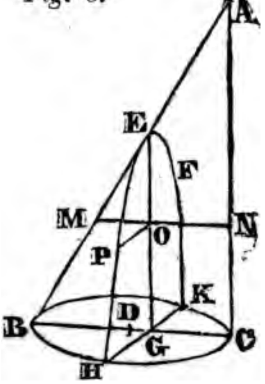
Fig. 2.



MOE and BGE, LON and LGC, MO is to BG, as EO is to EG, and ON to GC, as LO to LG; therefore the ratios of the rectangle MON to BGC and of EOL to EGL, being compounded of these equal ratios (23. 6 *Eu.*), are equal (22. 5 *Eu.*); therefore the square of PO is to the square of HG, as the rectangle EOL is to the rectangle EGL (11. 5 *Eu*). Let a hyperbola be described with the diameter EL, its vertices being E and L, and the ordinate HG (85. 1 *Sup.*), and, because the square of PO is to the square of HG, as the rectangle EOL is to the rectangle EGL, the point P is in the hyperbola (*Cor.* 3. 40. 1); in like manner it may be shewn, that any other point in the section HEK is in the hyperbola, and therefore that section is a hyperbola, of which EL is a diameter.

Thirdly. Let EG be parallel to a right line AC drawn from the vertex A of the cone to the circumference of its base, let H and K be the points in which HK meets that circumference, from any point P in the section HEK draw PO parallel to HK, meeting EG in O, and through O draw MN parallel to BC; as before, the plain by PO and MN is parallel to that by HG and BC, or to the base of the cone, and the angle POM equal to HGB, and therefore a right angle, therefore the square of PO is equal to the rectangle MON (3 and 35. 3 *Eu*). In like manner it may be shewn, that the square of HG is equal to the rectangle BGC; therefore the square of PO is to the square of HG, as the rectangle MON is to the rectangle BGC (*Cor.* 1. 7. 5 *Eu.*), or, because of the equals ON and GC (34. 1 *Eu.*), as MO is to BG (1. 6 *Eu.*), or, which is, because of the equiangular triangles MOE and BGE, equal (4. 6 and 16. 5 *Eu.*), as EO is to EG. Let a parabola be described with the diameter EG, its vertex being E, and ordinate HG (85. 1 *Sup.*), and because the square of PO is to the square of HG, as EO is to EG, the point P is in the parabola (*Cor.* 3. 40. 1 *Sup.*); in like manner it may be shewn, that any other point in the section HEK is in the parabola, and therefore that section is a parabola, of which EG is a diameter.

Fig. 3.



Scholium. From the five preceding propositions it appears, that the figures formed by the intersection of a plain with a conical surface, are the same, as those defined in Def. 1. 1 Sup.

ELEMENTS OF SPHERICAL TRIGONOMETRY.

NOTE.—*In subsequent citations, Sph. Tr. denotes, Spherical Trigonometry.*

DEFINITIONS.

1. *A great circle of a sphere,* is one, whose plain passes through the centre of the sphere.

2. The *pole* of a circle of a sphere, is a point in the surface of the sphere, from which all right lines drawn to the circumference of the circle are equal.

3. *A spherical angle,* is an angle on the surface of a sphere, contained by the arches of two great circles which meet each other; and is the same, as that which the plains of these circles make with each other; being the angle contained by two right lines, drawn in the plains of these circles, perpendicular to the line of common section, from the same point therein.

4. *A spherical triangle,* is a figure on the surface of a sphere, comprehended by the arches of three great circles.

PROP. I. THEOR.

All great circles of the same sphere are equal; and any two of them bisect each other.

Their radiuses being equal, as being each of them equal to the radius of the sphere, the circles are equal (*Cor.* 2. 24. 3 *Eu*).

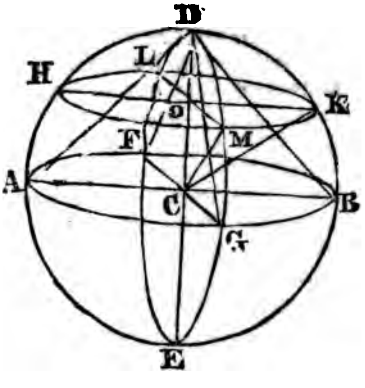
And since any two of them have the same centre, namely, the centre of the sphere, their common section is a diameter of both, and therefore bisects both (24. 3 *Eu*).

Cor. 1. The arches of two great circles of a sphere, being less than semicircles, do not contain a space; for, by this proposition, they only meet in opposite points of the circles.

Cor. 2. Two semicircles, meeting each other in opposite points, form equal angles at these opposite points, as is manifest from Def. 3. Sph. Tr. the plains of the circles forming these angles being the same.

PROP. II. THEOP.

The arch of a great circle, between the circumference of another great circle, and its pole, is a quadrant.



Let DG be an arch of a great circle, between the circumference of another great circle AGB, and its pole D; the arch DG is a quadrant.

Let the circle, of which DG is an arch, meet the circle AGB again in F, and let FG be the common section of the plains of these circles, which, because the circles bisect each other (1. *Sph. Tr.*), passes through the centre C; join DF and DG, which are equal (*Hyp. and Def.* 2 *Sph. Tr.*), therefore the arches DF and DG are equal (28. 3 *Eu.*); whence, FDG being a semicircle (1 *Sph. Tr.*); the angle DG is a quadrant (*Def.* 3. *Pl. Tr*).

Cor. 1. The point [D], wherein a right line [CD], drawn from the centre [C] of a sphere, perpendicular to the plain of a great circle [AGB], meets the surface of the sphere, is a pole of the circle.

To any points G and B in the circumference AGB, draw CG, CB, DG and DB, and because, in the triangles DCG and DCB, the side DC is common, the angles at C equal, being right (*Hyp. and Def.* 3. 2 *Sup.*), and CG equal to CB (*Def.* 10. 1 *Eu.*), DG and DB are equal (4. 1 *Eu.*); in like manner it may be proved, that all right lines drawn from D to the circumference of the circle AGB are equal; therefore D is the pole of that circle (*Def.* 2. *Sph. Tr*).

Cor. 2. A right line [DC], drawn from the pole [D] of any great circle [AGB] to the centre [C] of the sphere, is perpendicular to the plain of that circle.

Having drawn any two diameters ACB and FCG of the circle AGB, and joined DA and DB, because in the triangles DCA and DCB, DA is equal to DB (*Hyp. and Def.* 2 *Sph. Tr.*), AC to CB (*Def.* 10. 1 *Eu.*), and DC common, the angles DCA and DCB are equal (8. 1 *Eu.*), and so DC perpendicular to AB (*Def.* 20. 1 *Eu.*); in like manner it may be proved, that DC is perpendicular to FG; it is therefore perpendicular to the plain passing by AB and FG (2. 2 *Sup.*), or the plain of the circle AGB.

Cor. 3. The point [D], wherein a right line [OD] drawn from the centre [O] of a less circle [HMK] of a sphere, perpendicular to the plain of the circle, meets the surface of the sphere, is a pole of the circle.

Right lines being supposed to be drawn from the points O and D, to any two points M and K in the circumference HMK, the demonstration is similar to that in the 1st cor. to this prop.

Cor. 4. A right line [DO], drawn from the pole [D] of any less circle [HMK], to its centre [O], is perpendicular to the plain of the circle.

Any two diameters HOK and LOM of the circle HMK being drawn, and right lines being supposed to be drawn from D to H and K, the demonstration is similar to that in the 2d cor. to this proposition.

Cor. 5. Every circle of a sphere has two poles, one to each side of its plain, namely, the extremities of the diameter of the sphere, which is perpendicular to the plain.

Cor. 6 A great circle, which is at right angles to another great circle, passes through its poles; for a right line, drawn through the centre of the sphere, at right angles to the latter circle, is in the plain of the former (*Def.* 3. 2 *Sup. and Def.* 3. *Sph. Tr.*); whence, the poles of the latter being in that right line (*Cor.* 1 *to this prop.*), the former circle passes through these poles.

Cor. 7. A great circle, which passes through the poles of another, is at right angles to that other; for right lines drawn from these poles to the centre of the sphere, are perpendicular to the plain of the latter circle (*Cor.* 2 *of this prop.*), and therefore make right angles with a right line, drawn from that centre, in the plain of the former circle, perpendicular to the common section of the plains of the two circles (*Def.* 3. 2 *Sup.*), which angles are the measures of those, which these circles make with each other (*Def.* 3 *Sph. Tr*).

PROP. III. THEOR.

If two great circles (DG and DB, see fig. to prec. prop.), meeting a third (GB), intersect each other in the pole (D) of that third; the segment (GB) of the third circle between the other two, is the measure of the spherical angle, which the same two circles make with each other.

From the centre C of the sphere draw CD, CG and CB, and since D is the pole of the circle GB (*Hyp.*), the arches DG and DB are quadrants (2. *Sph. Tr.*), and therefore the angles DCG and DCB are right angles (*Cor.* 1. 33. 6 *Eu.*), and of course the angle GCB, and therefore the arch GB, which is the measure of it, is the measure of the spherical angle GDB (*Def.* 3 *Sph. Tr*).

Cor. If two arches of great circles [DG and DB], drawn from the same point [D], be each of them quadrants, their intersection [D] is the pole of the great circle [GB], which passes through their other extremes [G and B].

For, CD, CG and CB being drawn from the centre C of the sphere; because the arches DG and DB are quadrants, the angles DCG and DCB are right angles, therefore DC is perpendicular to the plain GCB, being the plain of the circle AGB (2. 2 *Sup.*), of which circle therefore D is the pole (*Cor.* 1. 2 *Sph. Tr*).

PROP. IV. THEOR.

If one great circle of a sphere meet another, the adjacent angles are together equal to two right angles.

PROP. V. THEOR.

If two great circles of a sphere intersect each other, the opposite angles are equal.

This proposition and the preceding, are demonstrated in like manner, as the corresponding properties of right lines, in the 13th and 15th prop. of the 1st book of Euclid's Elements.

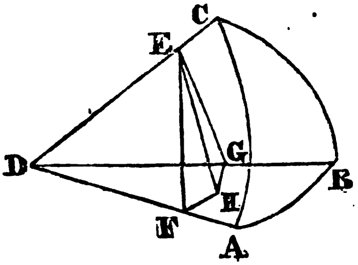
PROP. VI. THEOR.

Any side of a spherical triangle is less than a semicircle.

For if any side of a spherical triangle were equal to a semicircle, it would meet another side at both extremes (1 *Sph. Tr.*), and would therefore leave no space for a third side; a like absurdity, would follow, if any side were supposed to be greater than a semicircle.

PROP VII. THEOR.

The angles (CAB and CBA), at the base (AB) of an isosceles spherical triangle (ABC), are equal.



Let D be the centre of the sphere; join DA, DB and DC, and from any point E in the right line DC drawn to the vertex C of the triangle, let fall perpendiculars EF and EG on those DA and DB drawn to the extremes A and B of its base; from F and G, in the plain DAB, draw FH and GH at right angles to DA and DB, meeting each other in H, and join EH.

Because the angles DFE and DFH are right angles (*Constr.*), the angle EFH is the angle which the plains DAB and DAC make with each other, being the measure of the spherical angle CAB (*Def.* 3. *Sph. Tr.*); for the same reason, the angle EGH is the measure of the spherical angle CBA.

And because DF is perpendicular to each of the right lines EF and FH (*Constr.*), it is perpendicular to the plain EFH (2. 2 *Sup.*), therefore every plain passing by DF, and therefore the plain DAB, is perpendicular to the plain EFH (9. 2 *Sup.*); for the same reason, the plain DAB is perpendicular to the plain EGH; therefore the common section EH of the plains EFH and EGH is perpendicular to the plain DAB (10. 2 *Sup.*), and of course the angles EHF and EHG are right angles (*Def.* 3. 2 *Sup*).

And since the arches CA and CB are equal (*Hyp.*), the angles EDF and EDG, of which these arches are the measures (*Def.* 2 *Pl. Tr.*), are also equal; whence, the triangles DEF and DEG having the angles at F and G right, and DE common, EF is equal to EG (26. 1 *Eu.*); whence, the triangles EFH and EGH having the angles at H right, and EH common, the angles EFH and EGH are equal (*Cor.* 7. 6 *Eu.*); but it has been above shewn, that the angles EFH and EGH are the measures of the spherical angles CAB and CBA, therefore the spherical angles CAB and CBA, at the base of the isosceles triangle ABC, are equal.

PROP. VIII. THEOR.

If two angles (CAB and CBA, see fig. to the prec. prop.), of a spherical triangle (ABC), be equal, the sides (CB and CA), opposite to them, are equal.

The same construction, as in the preceding proposition, considering the side AB of the spherical triangle between the equal angles, as base, remaining, the angles EFH and EGH, may, as in that proposition, be shewn to be the measures of the spherical angles CAB and CBA, which spherical angles being equal (*Hyp.*), the angles EFH and EGH are equal, and the angles EHF and EHG may, as in the same proposition, be shewn to be right; whence, the triangles HEF and HEG having EH common, EF is equal to EG (26. 1 *Eu.*); and therefore the triangles FED and GED having the angles at F and G right (*Constr.*), and DE common, the angles FDE and GDE are equal (*Cor.* 7. 6 *Eu.*); and therefore the arches AC and BC, which are the measures of them (*Def.* 2. *Pl. Tr.*), are also equal.

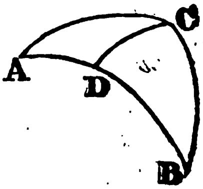
PROP. IX. THEOR.

Any two sides (as AB and BC, see figure to the seventh proposition), of a spherical triangle (ABC), are together greater than the third (AC.)

Let D be the centre of the sphere, and join DA, DB and DC. Of the three plain angles which contain the solid angle D, the two ADB and BDC are together greater than the third ADC (17. 2 *Sup.*), and the sides AB, BC and AC of the spherical triangle ABC are the measures of the angles ADB, BDC and ADC (*Def.* 2 *Pl. Tr*), therefore the sides AB and BC of the spherical triangle ABC are together greater than the third AC.

PROP. X. THEOR.

In any spherical triangle (ACB) the greater (ACB) of two unequal angles (ACB and CAB) is subtended by the greater side (AB); and the greater (AB) of two unequal sides (AB and CB) is subtended by the greater angle (ACB).

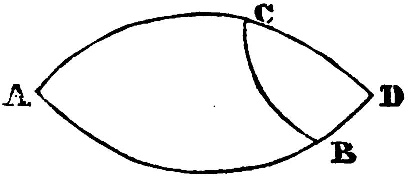


Part 1. From the greater angle ACB take a part ACD equal to the less A, and the side CD is equal to AD (8. *Sph. Tr.*); adding to each DB, the whole AB is equal to CD and DB together, but CD and DB together are greater than CB (9. *Sph. Tr.*), therefore AB is greater than CB.

Part 2. If AB be greater than BC, the angle ACB is greater than the angle A; for the angle ACB is not equal to A, for if it were, the side AB would be equal to BC (8. *Sph. Tr.*), contrary to the supposition; and the angle ACB is not less than A, for if it were AB would be less than BC (*by part* 1), which is also contrary to the supposition; therefore the angle ACB being neither equal to nor less than A, is greater than it.

PROP. XI. THEOR.

The three sides (AB, BC and AC), of a spherical triangle (ABC), are together less than a whole circle.



Produce AB and AC to meet each other in D; the arches ABD and ACD are each of them semicircles (1 *Sph. Tr.*); but in the triangle BDC, the side BC is less than BD and DC together (9. *Sph. Tr.*). adding to each the arches AB and AC, the three sides AB, BC and AC of the triangle ABC, are together less than the two arches ABD and ACD (*Ax.* 4. 1 *Eu.*), or, which is equal, than a whole circle.

Otherwise, see fig. to prop. 7.

Let D be the centre of the sphere, on which the triangle ABC is formed; join DA, DB and DC; and, because the three plain angles ADB, BDC and ADC, which contain the solid angle D, are together less than four right angles (18. 2 *Sup.*), the three sides AB, BC and AC, which are the measures of them (*Def.* 2 *Pl. Tr.*), are together less than a whole circle.

PROP. XII. THEOR.

In any spherical triangle (ABC, see fig. to prec. prop.), according as the sum of the legs (AC and CB), are equal to, or greater or less than, a semicircle, the sum of the angles (A and ABC) at the base, are equal to, or greater or less than, two right angles.

Let AB and AC be produced to meet in D.

Part 1. If AC and CB be together equal to a semicircle or to ACD, taking from each the common arch AC, the arches CB and CD are equal (*Ax.* 3. 1 *Eu.*), therefore the angle CBD is equal to the angle CDB (7. *Sph. Tr.*), or, which is equal (*Cor.* 2. 1 *Sph. Tr.*), CAB; adding to each the angle CBA, the two angles at the base CAB and CBA together are equal to the angles CBD and CBA together, or which is equal (4. *Sph. Tr.*), to two right angles.

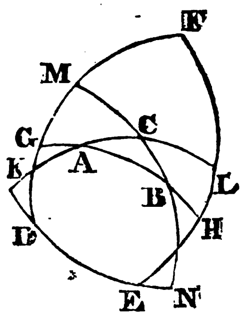
Part 2. If AC and CB together, be greater than a semicircle, or than ACD, CB is greater than CD; and therefore the angle D or A is greater than the angle CBD (10. *Sph. Tr.*); adding to each CBA, the angles CAB and CBA together, are greater than CBD and CBA together, or than two right angles.

Part 3. In like manner, if AC and CB together, be less than a semicircle, it may be shewn, that the angle D or A is less than CBD, and of course, the angles A and CBA together, less than CBD and CBA together, or than two right angles.

Cor. It follows, that, according as the sum of the sides [AC and CB] is equal to, or greater or less than, a semicircle, either angle at the base [as A] is equal to, or greater or less than, the external angle [CBD] at the other extreme of the base.

PROP. XIII. THEOR.

If a spherical triangle (DEF), be formed by great circles, joining the poles (F, D and E), of the sides (AB, BC and AC), of another spherical triangle (ABC); the sides of the former triangle (DEF), are complements of the measures of the opposite angles of the latter, to semicircles; and the measures of the angles of the former, are complements of the opposite sides of the latter, to semicircles.



Let AB, produced as necessary, meet DF and EF in G and H; BC meet DF and DE in M and N; and AC meet ED and EF in K and L.

Because F is the pole of the arch GAH (*Hyp.*), an arch drawn from F to B is a quadrant (2 *Sph. Tr.*), and, because D is the pole of the arch MCN (*Hyp.*), an arch drawn from D to B is a quadrant (2 *Sph. Tr.*); whence, DF being less than a semicircle (6 *Sph. Tr*), B is the pole of the arch DGMF (*Cor.* 3. *Sph. Tr.*), therefore BG and BM are quadrants (2 *Sph. Tr.*), and of course GM is the measure of the angle CBA (3. *Sph. Tr*). In like manner it may be shewn, that LH is the measure of the angle CAB; and KN of the angle ACB.

And, because F is the pole of the arch GAH (*Hyp.*), the arch FG is a quadrant (2 *Sph. Tr.*), and because D is the pole of the circle MCN (*Hyp.*), the arch DM is a quadrant (2 *Sph. Tr.*), therefore FG and DM together, or DF and GM together, are equal to a semicircle, and so the side FD, which is opposite the angle CBA, is the complement of GM, which is above shewn to be the measure of the angle CBA, to a simicircle. In like manner it may be shewn, that FE is the complement of the measure of the angle CAB, and DE, of the measure of the angle ACB, to semicircles.

And, because BM and CN are quadrants, these arches together, or MN and CB together, are equal to a semicircle; and therefore MN, the measure of the angle FDE, is the complement of the opposite side CB of the triangle ACB, to a semicircle. In like manner it may be demonstrated, that KL the measure of the angle DEF, is the complement of the side AC, and GH the measure of the angle F, of the side AB, to a semicircle.

Scholium. Although, in this proposition, the word complement is used in its customary meaning; yet the complement of an arch or angle, to a semicircle or two right angles, is also called its supplement, as mentioned in Def. 5. Pl. Tr.; and therefore one of the triangles mentioned in this proposition, is said to be *supplemental* of the other.

PROP. XIV. THEOR.

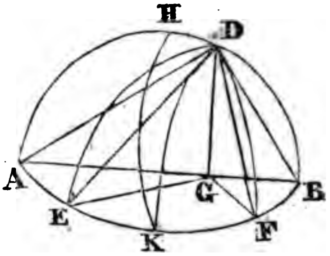
The three angles of a spherical triangle (ABC, see fig. to the preceding proposition), are greater than two right angles, and less than six.

For the three measures of the angles of the triangle ABC, together with the three sides of its supplemental triangle DEF, are equal to three semicircles (13. *Sph. Tr*); but the three sides of the triangle DEF are less than two semicircles (11. *Sph. Tr.*); therefore the three measures of the angles of the triangle ABC, are greater than a semicircle, and of course these three angles are together greater than two right angles.

And these three angles are less than six right angles, for the internal and external angles together of the triangle ABC, are equal to six right angles (4. *Sph. Tr.*), therefore the three internal angles are less than six right angles.

PROP. XV. THEOR.

Of all arches of great circles (DA, DE, DF and DB), which can be drawn to any great circle (AEB) of a sphere, from any point (D) which is not its pole, the greatest is that (DA) which passes through the pole (H), and the continuation of it (DB) is the least; and of others (DE and DF), that (DE) which is nearer to the greatest, is greater than the more remote (DF); and the angles (DEA and DEB) which any of them (as DE), which does not pass through its pole (H), makes with it, are unequal, that (DEA) which is towards the pole, being the greater.



Let AB be the common section of the circles ADB and AEB, from D draw DG perpendicular to AB, and join GE, GF, DA, DE, DF and DB.

Because AB is a diameter of the sphere (1. *Sph. Tr.*), its middle point is the centre of the same, and therefore a right line drawn thereto, from the pole H of the circle AEB, is perpendicular to the plain AEB (*Cor.* 2. 2 *Sph. Tr.*), and therefore to AB (*Def.* 3. 2 *Sup.*), and therefore, DG being perpendicular to AB (*Constr.*), parallel to DG (28. 1 *Eu.*); whence, the right line so drawn from H to the centre being perpendicnlar to the plain AEB, the right line DG is perpendicular to the same plain (5. 2 *Sup.*),, and so the angles DGA, DGE, DGF and DGB are right angles (*Def.* 3. 2 *Sup*).

And GA is the greatest, and GB the least, of all right lines, which can be drawn from G to the circumference AEB, and the right line GE greater than GF (7. 3 *Eu.*); whence, the triangles DGA, DGE, DGF and DGB being right angled at G, and having the side DG common, the square of DG and GA together, or, which is equal (47. 1 *Eu.*), the square of DA, is greater than the squares of DG and GE together, or which is equal (*by the same*), the souare of DE; therefore the right line DA is greater than DE. In like manner it may be shewn, that the right line DE is greater than DF, and DF than DB; whence it follows, that the arch DHA is the greatest, and DB the least, of all arches of great circles, which can be drawn from D to the great circle AEB, and the arch DE, which is nearer to DHA, greater than the arch DF which is more remote.

And an arch of a great circle being supposed to be drawn from H to E, the spherical angle HEA is a right angle (*Cor.* 7. 2 *Sph. Tr.*), therefore the spherical angle DEA is greater, and of course (4. *Sph. Tr.*), DEB less than a right angle. In like manner it may be proved, that the spherical angle DFA is greater, and DFB less than a right angle.

Cor. 1. Hence it follows, that to any great circle [AEB] of a sphere, from any point [D] which is not its pole, but two perpendiculars [DHA and DB] can be drawn, one [DHA] passing through its pole [H], and the other [DB] the continuation thereof, their incidences being at the interval of a semicircle from each other.

Cor. 2. If to a great circle [AEB] of a sphere, from any point [D] without it which is not its pole, an arch [DK] of a great circle be drawn, to the point [K] in which an arch [AEB], intercepted between the perpendiculars [DHA and DB] let fall from that point on the great circle, is bisected; the arch [DK] so drawn is a quadrant.

For the angle KBD being a right angle, and KB a quadrant, K is the pole of the circle BD (*Cor.* 6. 2 *and prop.* 2 *Sph. Tr.*), and therefore DK is a quadrant (2. *Sph. Tr*).

PROP. XVI. THEOR.

The legs containing the right angle of a right angled spherical triangle, are of the same affection, as the opposite angles. That is, according to the legs are equal to, or greater or less than, quadrants, the angles opposite to them are equal to, or greater or less than, right angles.

Part 1. Let the spherical triangle HAK (see fig. to the prec. prop.), right angled at A, have the leg AH equal to a quadrant, K being supposed to be any where in the circle AEB, H is the pole of the arch AK (*Cor.* 6. 2 *and prop.* 2 *Sph. Tr.*), therefore the angle HKA is a right angle (*Cor.* 7. 2 *Sph. Tr*).

Part 2. Let the spherical triangle DEA or DFA, right angled at A, have a leg, as AHD, greater than a quadrant, and the angle DEA or DFA, opposite that leg, is obtuse (15. *Sph. Tr*).

Part 3. Let the spherical triangle DFB or DEB, right angled at B, have a leg, as BD, less than a quadrant, and the angle DFB or DEB, opposite that leg, is acute (15 *Sph. Tr*).

PROP. XVII. THEOR.

If two legs of a right angled spherical triangle be of the same affection, (and consequently by the preceding proposition, the angles opposite to them) ; the hypothenuse is less than a quadrant ; unless both the legs be quadrants, in which case the hypothenuse is a quadrant ; if they be of different affections, the hypothenuse is greater than a quadrant.

Part 1. If in the spherical triangle ADF, see fig. to proposition 15, right angled at A, the legs AD and AF be both greater than quadrants, or in the triangle BDF, right angled at B, the legs BD and BF be both less than quadrants ; the hypothenuse DF is less than a quadrant.

For the semicircle AEB being bisected in K, the arch of a great circle DK being drawn is a quadrant (*Cor.* 2. 15 *Sph. Tr.*), and DF is less than DK (15. *Sph. Tr.*), therefore DF is less than a quadrant.

Part 2. If in the spherical triangle AHK, right angled at A, the legs AH and AK be quadrants ; the third side HK is a quadrant.

For A is the pole of the circle HK (*Cor.* 3. *Sph. Tr.*) ; whence, the spherical angle KAH being a right angle (*Hyp.*), the arch HK is a quadrant (3 *Sph. Tr*).

Part 3. If in the spherical triangle ADE, right angled at A, one of the legs AD be greater, and the other AE less than a quadrant ; DE is greater than a quadrant.

For the arch DK being drawn as in part 1, is a quadrant ; and DE is greater than DK (15 *Sph. Tr.*) ; therefore DE is greater than a quadrant.

PROP. XVIII. THEOR.

According as the hypothenuse of a right angled spherical triangle is greater or less than a quadrant ; the legs, (and consequently by 16. *Sph. Tr. the oblique angles), are of different affections, or of the same.*

Part 1. If the hypothenuse be greater than a quadrant, the legs are of different affections.

For if the legs were of the same affection, the hypothenuse would not be greater than a quadrant (17. *Sph. Tr.*), contrary to the supposition.

Part 2. If the hypothenuse be less than a quadrant, the legs are of the same affection.

For if they were of different affections, the hypothenuse would be greater than a quadrant (17 *Sph. Tr.*), contrary to the supposition.

PROP. XIX. THEOR.

If, in any spherical triangle, (ABC, see figure 1 and 2), the perpendicular (CD) let fall on the base (AB) from the opposite angle, fall within the triangle (as in figure 1); the angles at the base are of the same affection: if the perpendicular fall without the triangle (as in fig. 2); the angles at the base are of different affections.

Fig. 1.

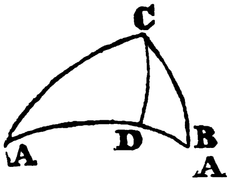
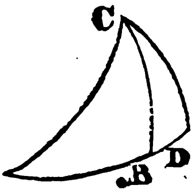


Fig. 2.



Part 1. *Fig.* 1. Since the triangles ACD and BCD are both right angled at D; the angles A and B are of the same affection with CD (16 *Sph. Tr.*), and therefore with each other.

Part 2. When the perpendicular CD falls without the triangle, as in fig. 2, the angles A and CBD are of the same affection with CD (16 *Sph. Tr.*), and therefore with each other, and the angles CBA, and CBD are of different affections (4. *Sph. Tr.*); therefore the angles A and ABC are of different affections

Scholium. In part 2, it is supposed, that CD is not equal to a quadrant, for if it were, C would be the pole of the arch ABD (*Cor.* 6. 2 *and prop.* 2 *Sph. Tr.*), and all the angles CAB, CBA and CBD would be right angles (*Cor.* 7. 2 *Sph. Tr*). A like observation is applicable to the next proposition, the case being excepted, when the vertex of the triangle is the pole of the base.

PROP. XX. THEOR.

If the angles at the base of a spherical triangle, be of the same affection, the perpendicular let fall thereon from the opposite angles, falls within the triangle; if of different affections, the perpendicular falls without.

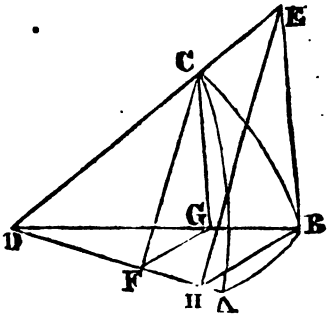
Part 1. If the angles at the base of a spherical triangle be of the same affection, the perpendicular let fall thereon from the opposite angle, falls within the triangle, as CD on AB in figure 1 preceding proposition; for if it did not fall within, the angles A and B at the base would be of different affections (19 *Sph. Tr.*), contrary to the supposition.

Part 2. If the angles at the base be of different affections, the perpendicular let fall thereon from the opposite angle, falls without the triangle, as CD on AB produced in fig. 2; for if it did not fall without, the angles at the base would be of the same affection (19 *Sph. Tr.*), contrary to the supposition.

PROP. XXI. THEOR.

In a right angled spherical triangle (ABC, right angled at B), the rectangle under radius and the sine of either leg (BC), is equal to the rectangle under the sine of the angle (BAC) opposite that leg, and the sine of the hypothenuse (AC.)

Let D be the centre of the sphere; join DA, DB and DC, and draw BH perpendicular to DA; BH is the sine of the arch AB, the semidiameter DA of the sphere being radius (*Def.* 6 *Pl.* Tr.); from H, draw HE, in the plain DAC, perpendicular to DA, meeting DC in E, and join BE; from C draw CF perpendicular to DA; from F in the plain DAB, the right line FG at right angles to DA; and join CG: CF is the sine of the arch AC to the same radius.



And since DH is perpendicular to both HB and HE, it is perpendicular to the plain EHB (2. 2 *Sup.*), therefore the plain DAB which passes by DH is perpendicular to the plain EHB (9. 2 *Sup.*), and so the plain EHB is perpendicular to the plain DAB, as is manifest from Def. 4. 2 Snp.; but the plain DBC or DBE is, because of the spherical right angle at B, perpendicular to the same plain DAB (*Def.* 3. *Sph. Tr.*); therefore BE the common section of the plains HBE and DBE is perpendicular to the plain DAB (10. 2 *Sup.*), and EBD and EBH are right angles (*Def.* 3. 2 *Sup.*); therefore BE is a tangent of the arch BC to the same radius (*Def.* 8 *Pl. Tr*).

In like manner CG may be shewn to be at right angles to the plain DAB, therefore CGD and CGF are right angles, and CG is the sine of the arch CB to the same radius.

And in the triangle CGF right angled at G, CG is to CF, as the sine of the angle CFG, or (*Def.* 3 *Sph. Tr.*), of the spherical angle CAB, is to radius (1. *Pl. Tr.*); but, as has been just shewn, CG is to CF, as the sine of CB is to the sine of CA; therefore (11. 5 *Eu.*), the sine of the angle CAB is to radius, as the sine of CB is to the sine of CA; and therefore the rectangle under radius and the sine of CB, is equal to the rectangle under the sine of CAB and the sine of CA (16. 6 *Eu*).

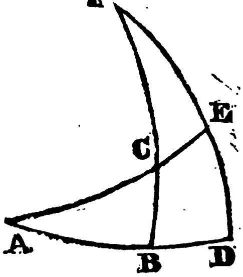
PROP. XXII. THEOR.

In a right angled spherical triangle (ABC, right angled at B, see fig. to prec. prop.), the rectangle under radius and the tangent of either leg (BC), is equal to the rectangle under the tangent of the angle (CAB) opposite that leg, and the sine of the other leg (AB).

The same construction remaining, as in the preceding proposition, in the triangle EBH right angled at B, BE is to BH, as the tangent of the angle EHB, or (*Def.* 3 *Sph Tr.*) of the spherical angle CAB, is to radius (2. *Pl. Tr.*); but, as has been shewn in the preceding prop. BE is to BH, as the tangent of BC is to the sine of BA; therefore (11. 5 *Eu.*), the tangent of the angle CAB is to radius, as the tangent of BC is to the sine of BA, and therefore the rectangle under radius and the tangent of CB, is equal to the rectangle under the tangent of CAB and the sine of AB (16. 6 *Eu*).

PROP. XXIII. THEOR.

If in a right angled spherical triangle, either of the legs, the complement of the hypothenuse, or of either of the angles, except the right angle, be taken as a middle part; the rectangle under radius and the sine of that middle part, is equal to the rectangle under the cosines of those two of the other parts, which, the right angle being excluded, are remote from that middle part.



Let ABC be a right angled spherical triangle, right angled at B and let either of the legs, AB or BC, the complement of the hypothenuse AC, or of either of the angles except the right angle, namely, of the angle A or ACB, be taken as a middle part; the rectangle under radius and the sine of that middle part, is equal to the rectangle under the cosines of those two of the other parts, which, the right angle being excluded, are remote from that middle part.

Let the great circle DF be described of which A is the pole, meeting AB, AC and BC produced in D, E and F; and since the great circle AD passes through the pole A of the great circle DE, the angle D is a right angle (*Cor.* 7. 2 *Sph. Tr*), and therefore DE passes through the pole of the arch ABD (*Cor.* 6. 2 *Sph. Tr.*); and, because the angle ABC is a right angle, BCF passes also through the pole of ABD (*by the same*); therefore the arches AD, AE, FB and FD are quadrants (2 *Sph. Tr.*), and the angle CEF is a right angle (*Cor.* 7. 2 *Sph. Tr.*); therefore in the triangle CEF right angled at E, CE is the complement of the hypothenuse AC of the triangle ABC; also FC is the complement of BC, EF of ED, the measure of the angle A (3 *Sph. Tr.*), and BD, which is the measure of the angle F (*by the same*), is the complement of AB. These things being premised.

Case 1. When either leg, as AB, is the middle part, and the complements of AC and ACB, the remote parts.

The rectangle under radius and the sine of AB, is equal to the rectangle under the sines of the angle ACB and of the hypothenuse AC (21 *Sph. Tr*).

Case 2. When the complement of either of the oblique angles, as A, is the middle part, and BC and the complement of ACB, the remote parts.

In the triangle CEF, right angled at E, the rectangle under radius and the sine of EF is, by 21 Sph. Tr., equal to the rectangle under the sine of ECF, or its equal (5. *Sph. Tr.*), ACB, and the sine of CF; and therefore, substituting the equals in the triangle ABC, the rectangle under radius and the cosine of A, is equal to the rectangle under the sine of the angle ACB and the cosine of BC.

Case 3. When the complement of the hypothenuse AC is the middle part, and the legs AB and BC, the remote parts.

In the triangle CEF, right angled at E, the rectangle under radius and the sine of CE is, by 21. Sph. Tr., equal to the rectangle under the sines of F and FC; and therefore, substituting the equals in the triangle ABC, the rectangle under radius and the cosine of AC, is equal to the rectangle under the cosines of AB and AC.

PROP. XXIV. THEOR.

The same things being supposed, the rectangle under radius and the sine of the middle part, is equal to the rectangle under the tangents of the parts, which, the right angle being excluded, are adjacent to the middle part.

The same construction remaining, as in the preceding proposition.

Case 1. When either leg, as AB, is the middle part, and BC and the complement of A, adjacent parts.

The rectangle under radius and the tangent of BC, is equal to the rectangle under the tangent of A and sine of A › (22 *Sph. Tr.*), therefore the sine of AB is to radius, as the tangent of BC is to the tangent of A (16. 6 *Eu.*); but radius is to the cotangent of A, as the tangent of A is to the radius (*Cor.* 4 *Def. Pl. Tr. and Theor.* 3. 15. 5 *Eu.*); therefore, by equality, the sine of AB is to the cotangent of A, as the tangent of BC is to radius (22. 5 *Eu.*); therefore the rectangle under radius and the sine of AB, is equal to the rectangle under the cotangent of A and the tangent of BC (16. 6 *Eu*).

Case 2. When the complement of either of the oblique angles, as A, is the middle part, and AB and the complement of AC, adjacent parts.

In the triangle CEF, right angled at E, the rectangle under radius and the tangent of CE, is equal to the rectangle under the tangent of F and the sine of EF (22 *Sph. Tr.*), therefore the sine of EF is to radius, as the tangent of CE is to the tangent of F (16. 6 *Eu.*); and therefore, substituting the equals in the triangle ABC, the cosine of A is to radius, as the cotangent of AC is to the cotangent of AB, or, tangents being reciprocally as their cotangents (*Cor.* 5 *Def. Pl. Tr.*), as the tangent of AB is to the tangent of AC; and radius is to the cotangent of AC, as the tangent of AC is to radius (*Cor.* 4 *Def. Pl. Tr. and Theor.* 3. 15. 5 *Eu.*); therefore, by equality, the cosine of A is to the cotangent of AC, as the tangent of AB is to radius (22. 5 *Eu.*; and therefore the rectangle under radius and the cosine of A, is equal to the rectangle under the cotangent of AC and the tangent of AB (16. 6 *Eu*).

Case 3. When the complement of the hypothenuse AC is the middle part, and the complements of the angles A and ACB, adjacent parts.

In the triangle CEF, right-angled at E, the rectangle under radius and the tangent of EF, is equal to the rectangle under the tangent of ECF and the sine of CE (22. *Sph. Tr.*), or substituting the equals in the triangle ABC, the rectangle under radius and the cotangent of A, is equal to the rectangle under the tangent of ACB and the cosine of AC; therefore the cosine of AC is to radius, as the cotangent of A is to the tangent of ACB (16. 6 *Eu.*); and radius is to the cotangent of ACB, as the tangent of ACB is to radius (*Cor.* 4 *Def. Pl. Tr. and Theor.* 3. 15. 5 *Eu.*); therefore, by equality, the cosine of AC is to the cotangent of ACB, as the cotangent of A is to radius (22. 5 *Eu.*), and of course the rectangle under radius and the cosine of AC, is equal to the rectangle under the cotangents of A and ACB (16. 6 *Eu*).

Scholium.—In any right angled spherical triangle, the five parts mentioned in this proposition and the preceding, namely, the two legs, and the complements of the hypothenuse and oblique angles, are called, *circular parts*; of which, any one being considered as a *middle part*, those, which, the right angle being excluded, are adjacent thereto, are called *adjacent parts*, and the other two, *remote parts*, as in these propositions; both the adjacent and remote parts being called by a common name, *extreme parts.*

PROP. XXV. THEOR.

If in an oblique anged spherical triangle (ABC), two right angled spherical triangles (ADC and BDC) be formed, by letting fall a perpendicular (CD), on any side (AB) considered as the base, from the opposite angle, and the perpendicular (CD) be assumed as the middle part in each of the right angled triangles; the rectangle under the cosines of the remote parts in one of the right angled triangles, is equal to that under the cosines of the remote parts in the other; and the rectangle under the tangents of the adjacent parts in one, is equal to that under the tangents of the adjacent parts in the other.

Fig. 1.

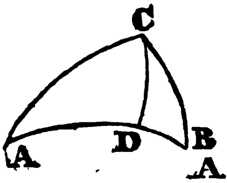
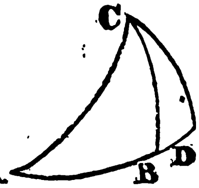


Fig. 2.



For the rectangles under the cosines of the remote parts in the triangle ADC, which are the complements of A and AC, and in the triangle BDC, which are the complements of CBD and BC, are each of them equal to the rectangle under radius and the sine of the middle part CD (23 *Sph. Tr.*), and therefore to each other.

And the rectangle under the tangents of the adjacent parts in the triangle ADC, which are AD and the complement of ACD, and in the triangle BDC, which are BD and the complement of BCD, are each of them equal to the rectangle under radius and the sine of the middle part CD (24. *Sph. Tr.*), and therefore to each other.

Cor. The tangents of the segments of the base [AD and DB] are to each other, as the tangents of the vertical angles [ACD and BCD].

For, by this proposition, the rectangle under the tangent of AD and cotangent of ACD is equal to the rectangle under the tangent of DB and cotangent of BCD; therefore the tangent of AD is to the tangent of DB, as the cotangent of BCD is to the cotangent of ACD (16. 6 *Eu.*), or, the tangents of any two arches or angles being reciprocally as their cotangents (*Cor.* 5 *Def. Pl. Tr.*), as the tangent of ACD is to the tangent of BCD.

PROP. XXVI. THEOR.

The same things being supposed, except that a middle part be so assumed in each of the right angled triangles, that the perpendicular may have a similar situation with respect to both these middle parts, and of course may be an extreme part of the same name, in both the right angled triangles; the sines of the middle parts are to each other, as the cosines of the remote, or as the tangents of the adjacent parts, which are peculiar to each of the right angled triangles, according as the perpendicular becomes a remote or an adjacent part.

Case 1. When the perpendicular CD, see fig. to prec. prop., is a remote part.

The rectangle under radius and the sine of the part assumed as a middle part in the triangle ADC, is equal to the rectangle under the cosines of CD and the other remote part in the same triangle (23. *Sph. Tr.*); and the like being true with respect to the triangle BDC; the rectangle under radius and the sine of the middle part in the triangle ADC, is to the rectangle under radius and the sine of the middle part in the triangle BDC, as rectangle under the cosines of CD and the other remote part in the triangle ADC, is to the rectangle under the cosines of CD and the other remote part in the triangle BDC (*Cor.* 1. 7. 5 *Eu.*); whence, the two first terms of these proportionals having a common side, namely, radius, and also the two latter, namely, the cosine of CD, by omitting these common sides, the sines of the middle parts are to each other, as the cosines of the remote parts which are peculiar to the triangles ADCand BDC (1. 6 and 11. 5 *Eu*).

Case 2. When the perpendicular CD is an adjacent part.

The rectangle under radius and the sine of the part assumed as a middle part in the triangle ADC, is equal to the rectangle under the tangents of CD and the other adjacent part in the same triangle (24 *Sph. Tr.*); and the like being true with respect to the triangle BDC; the rectangle under radius and the sine of the middle part in the triangle ADC, is to the rectangle under radius and the sine of the middle part in the triangle BDC, as the rectangle under the tangents of CD and the other adjacent part in the triangle ADC, is to the rectangle under the tangents of

CD and the other adjacent part in the triangle BDC (*Cor*. 1.7. 5 *Eu.*); whence, the two first terms of these proportionals having a common side, namely, radius, and also the two latter, namely, the tangent of CD, by omitting these common sides, the sines of the middle parts are to each other, as the tangents of the adjacent parts which are peculiar to the triangles ADC and BDC (1. 6 and 11. 5 *Eu*).

Cor. 1. If in a right angled spherical triangle [ABC, see fig. to prop. 25], a perpendicular [CD] be let fall on any side [AB] considered as the base, from the opposite angle; the cosines of angles at the base [A and ABC] are to each other, as the sines of the vertical angles [ACD and BCD].

In the right angled triangles ADC and BDC, the complements of A and CBD being assumed as middle parts, CD and the complements of the angles ACD and BCD are remote parts; therefore, by case 1 of this proposition, the cosine of A is to the cosine of CBD, or, by Cor. 1 Def. Pl. Tr., in fig. 2, of CBA, as the sine of ACD is to the sine of BCD.

Cor. 2. The same thing being supposed; the cosines of the sides [AC and BC] are to each other, as the cosines of the segments of the base [AD and BD].

In the same right angled triangles, assuming the complements of AC and CB as middle parts, CD and the segments of the base AD and BD are remote parts; therefore, by case 1 of this proposition, the cosines of AC and CB are to each other, as the cosines of AD and BD.

Cor. 3. The same thing being supposed; the sines of the segments of the base [AD and BD] are to each other, reciprocally as the tangents of the angles at the base [A and ABC].

In the same right angled triangles, assuming AD and BD as middle parts, CD and the complements of the angles A and CBD become adjacent parts; therefore, by case 2 of this proposition, the sines of AD and BD are to each other, as the cotangents of the angles A and CBD, or by Cor. 1 Def. Pl. Tr., in fig. 2, CBA, or, the tangents of any two angles being inversely as their cotangents (*Cor*. 5 *Def. Pl. Tr.*), reciprocally as the tangents of the same angles.

Cor. 4. The same thing being supposed; the cosines of the vertical angles [ACD and BCD] are to each other, reciprocally as the tangents of the sides [AC and BC].

In the same right angled triangles, the complements of the angles ACB and BCD being assumed as middle parts, CD and the complements of AC and CB are adjacent parts; therefore, by case 2 of this proposition, the cosines of ACB and BCD are to each other, as the cotangent of AC and CB, or, the tangents of any two arches being inversely as their cotangents (*Cor.* 5 *Def. Pl. Tr.*), reciprocally as the tangents of the same arches.

PROP. XXVII. THEOR.

The sines of the sides of spherical triangles, are to each other, as the sines of the opposite angles.

In the case of right angled spherical triangles, the proposition is manifest from 21 Sph. Tr., 16. 6 Eu. and Cor. 2 Def. Pl. Tr.

But if the spherical triangle be oblique angled, as ABC, see the figures to proposition 25, the sines of any two sides, as AC and BC, are to each other, as the sines of the opposite angles CBA and CAB.

Let fall on the third side AB, produced if necessary, the perpendicular CD, which perpendicular being assumed as a middle part in each of the right angled triangles ADC and BDC, the complements of AC and of the angle A become the remote parts in the former triangle, and the complements of BC and the angle DBC, the remote parts in the latter; therefore the rectangle under the sines of AC and the angle A, is equal to the rectangle under the sines of BC and the angle DBC (25 *Sph. Tr.*); therefore the sine of AC is to the sine of BC, as the sine of the angle DBC, or, which is equal in fig. 2 (*Cor.* 1 *Def. Pl. Tr.*), of ABC, is to the sine of the angle A (16. 6 *Eu*).

PROP. XXVIII. THEOR.

If on any side of a spherical triangle, considered as its base, a perpendicular be let fall from the opposite angle; the rectangle under the tangents of the half sum and half difference of the legs, is equal to the rectangle under the tangents of the half sum and half difference of the segments of the base, between its extremes and the perpendicular.

Fig. 1.

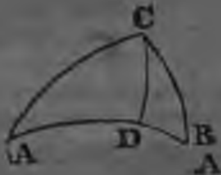
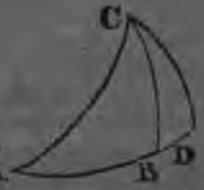


Fig. 2.



Let ABC be a spherical triangle, and CD a perpendicular let fall on the base AB from the opposite angle ACB; the rectangle under the tangents of the half sum and half difference of AC and BC, is equal to the rectangle under the tangents of the half sum and half difference of AD and BD.

For the cosine of AC is to the cosine of BC, as the cosine of AD is to the cosine BD (*Cor.* 2. 26 *Sph. Tr.*); therefore, by comparing the sums and differences of the terms, the sum of the cosines of AC and BC is to their difference, as the sum of the cosines of AD and BD is to their difference (17, 18 and 22. 5 *Eu.*); but the sum of the cosines of any two arches is to their difference, as the cotangent of half their sum is to the tangent of half their difference (5 *Pl. Tr.*); therefore, substituting the latter ratios for the former, the cotangent of half the sum of AC and BC is to the tangent of half their difference, as the cotangent of half the sum of AD and BD is to the tangent of half their difference; and, forming rectangles from the two first terms, with a common side of the tangent of the half sum of AC and BC, and from the two last, with a common side of the tangent of the half sum of AD and BD, the rectangle under the tangent and cotangent of the half sum of AC and BC, is to the rectangle under the tangents of the half sum and half difference of AC and BC, as the rectangle under the tangent and cotangent of the half sum of AD and BD, is to the rectangle under the tangents of the half sum and half difference of AD and BD (1. 6 and 11. 5 *Eu.*); but the first and third terms of these proportionals are equal, being each equal to the square of radius (*Cor.* 4 *Def. Pl. Tr. and* 16. 6 *Eu.*), therefore the second and fourth terms are equal (14. 5 *Eu.*), namely, the rectangles under the tangents of the half sum and half difference of AC and BC, and under the tangents of the half sum and half difference of AD and BD.

PROP. XXIX. THEOR.

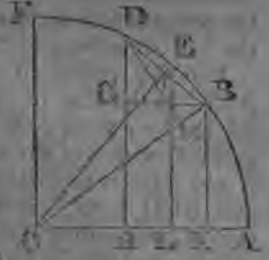
The same things being supposed, the rectangle under the tangents of the half sum and half difference of the angles at the base, is equal to the rectangle under the cotangent of the half sum and tangent of the half difference of the segments of the vertical angle, made by the legs with the perpendicular.

The same figure and construction remaining as in the preceding proposition, the rectangle under the tangents of the half sum and half difference of the angles A and ABC, is equal to the rectangle under the cotangent of the half sum and tangent of the half difference of the angles ACD and BCD.

For the cosine of the angle A is to the cosine of the angle ABC, as the sine of the angle ACD is to the sine of the angle BCD (*Cor.* 1. 26 *Sph. Tr.*); therefore, by comparing the sums and differences of the terms, the sum of the cosines of the angles A and ABC is to their difference, as the sum of the sines of ACD and BCD is to their difference (17. 18 and 22. 5 *Eu.*); but the sum of the cosines of the angles A and ABC is to their difference, as the cotangent of half their sum is to the tangent of half their difference (5 *Pl. Tr.*), and the sum of the sines of the angles ACD and BCD is to their difference, as the tangent of half their sum is to the tangent of half their difference (4 *Pl. Tr*); therefore, substituting the latter ratios for the former, the cotangent of the half sum of the angles A and ABC is to the tangent of half their difference, as the tangent of the half sum of the angles ACD and BCD is to the tangent of half their difference; and, forming rectangles from the two first terms, with a common side of the tangent of the half sum of the angles A and ABC, and from the two last, with a common side of the cotangent of the half sum of the angles ACD and BCD, the rectangle under the tangent and cotangent of the half sum of the angles A and ABC, is to the rectangle under the tangents of their half sum and half difference, as the rectangle under the tangent and cotangent of the half sum of the angles ACD and BCD, is to the rectangle under the cotangent of the half sum of these angles and the tangent of half their difference (1. 6 and 11. 5 *Eu.*); but the first and third of these four proportionals are equal, being each equal to the square of radius (*Cor.* 4 *Def. Pl. Tr. and* 16. 6 *Eu.*), therefore the second and fourth terms are equal (14. 5 *Eu.*), namely, the rectangles under the tangents of the half sum and half difference of the angles A and ABC, and under the cotangent of the half sum and tangent of the half difference of the angles ACD and BCD.

LEMMA I.

The rectangle under half the radius and the difference of the versed sines of any two arches, is equal to the rectangle under the sines of half the sum and half the difference of the same arches.



Let AB and AD be two unequal arches, and let their difference BD be bisected in E; AE is equal to half the sum, and DE to half the difference of those arches. Let C be the centre of the circle, and let ABF be taken equal to a quadrant; join CF, CE, CB, CA and DB, draw DH, EL and BK at right angles to CA, BG to DH, and DM to CB, and let DB meet CE in I. GB or HK is the difference of their versed sines HA and KA, and DI is the sine of half their difference. And since the triangles CLE and DGB are equiangular, because of the right angles at L and G, and the angle BDG at the circumference insisting on an arch equal to the two arches AB and AD taken together, and of course double to that AE which subtends the angle ECL at the centre, EL is to GB or HK, as CE is to DB (4. 6 and 16. 5 *Eu.*), or (15. 5 *Eu.*), as the half of CE or the half of radius, is to the half of DB, or, which is equal, to DI; therefore the rectangle under half the radius, and HK the difference of the versed sines of the arches AB and AD, is equal to the rectangle under EL the sine of the half sum, and DI the sine of the half difference of the same arches (16. 6 *Eu.*).

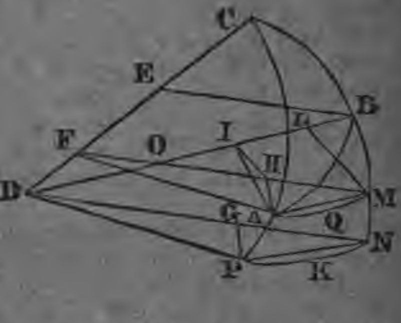
LEMMA II.

The rectangle under half the radius, and the versed sine (MB, see fig. to prec. prop.), or any arch (BD), is equal to the square of the sine (BI) of half the same arch.

For the triangles CBI and DMB are, because of the right angles at I and M, and the common angle at B, equiangular.— Therefore MB is to BD, as BI is to BC or radius (4. 6 *Eu.*), and, halving the consequents, MB is to the half of BD or to BI, as BI is to the half of radius (*Theor.* 2. 15. 5 *Eu.*); therefore the rectangle under half the radius and MB, is equal to the square of BI (17. 6 *Eu*).

PROP XXX. THEOR.

The rectangle under the sines of the legs of any spherical triangle, is to the rectangle under the sines of the excesses of the half sum of all the sides above each of the legs, in a duplicate ratio of radius to the sine of half the vertical angle.



Let ABC be a spherical triangle, of which AB is the base; the rectangle under the sines of the legs CA and CB, is to the rectangle under the sines of the excesses of the half sum of AB, BC and AC above CA and CB, in a duplicate ratio of radius to the sine of the vertical angle ACB.

Let AC be the greater of the legs AC and BC, and D the centre of the sphere. On CA produced, take CP equal to a quadrant, and on CB produced, take CM equal to CA, and CN equal to a quadrant. From the pole C let the arch of a great circle PKN and of the less one AQM be described, join DC, DB, DN and DP; let FM be the common section of the planes DCN and AQM, which let DB meet in O, and join FA; let fall the perpendiculars PG and AH on DN and FM, and OE on DC; let fall the perpendiculars IH and ML on DB, and join AI.

Because the plains DPN and FAM are perpendicular to the right line DC drawn from their pole C to the centre D of the sphere (*Cor.* 2 *and* 4. 2 *Sph. Tr. and* 19. 2 *Sup.*), they are perpendicular to the plain DCN (9. 2 *Sup.*): and because the plain DPN is perpendicular to the plain DCN, and PG perpendicular to their common section DN (*Constr.*), PG is perpendicular to a right line drawn from G in the plain DCN perpendicular to DN (*Def.* 4 *and* 5. 2 *Sup.*), and therefore to the plain DCN (2. 2 *Sup.*); in like manner, AH may be proved to be perpendicular to the plain DCN; whence, HI being perpendicular to DB (*Constr.*), DB is perpendicular to IA (*Cor.* 9. 2 *Sup.*); therefore IB is the versed sine of the arch AB, as LB is of the arch MB, the semidiameter of the sphere being radius (*Def.* 7. *Pl. Tr*).

And because the arches AM and PN are circular, having the centres at F and D in the right line DC (19. 2 *Sup.*), the triangles FAM and DPN are isosceles, and, because their vertical angles at F and D are equal (15 and 12. 2 *Sup.*), equiangular; and AH being perpendicular to FM and PG to DN (*Constr.*), the triangles into which these perpendiculars divide them are similar, therefore FM is to DN, as HM is to GN (4. 6 and 16 and 22. 5 *Eu.*); and, BE being drawn at right angles to DC, because the triangles DEB, HIO and MLO are equiangular, EB is to DB, as IL is to HM (4. 6 and 19. 5 *Eu.*), and compounding these two ratios, the rectangle under FM the sine of CM or CA, and EB the sine of CB, is to the rectangle under DN and DB, or the square of radius, as the rectangle under HM and IL is to the rectangle under HM and GN (23. 6 and 22. 5 *Eu.*), or, which is equal (1. 6 *Eu.*), as IL is to GN, or, which is equal (*by the same*), as the rectangle under the half of radius and IL, is to the rectangle under the half of radius and GN.

But the rectangle under the half of radius and IL, the difference of the versed sines of the arches AB and BM, is equal to the rectangle under the sines of the half sum and half difference of the arches AB and BM (*Lem.* 1 *to this prop.*); and because CA is equal to CM, the half sum of AB and BM is equal to the excess of the half sum of AB and AC above the half of BC, or, to the excess of the half sum of the sides AB, AC and BC above BC; and the half difference of AB and BM is equal to the excess of the half sum of AB and BC above the half of AC, or, to the excess of the half sum of the sides AB, BC and AC above AC; also the rectangle under the half of radius and GN, the versed sine of the arch PN, is equal to the square of the sine of half the same arch PN (*Lem.* 2 *to this prop.*), or, of half the angle ACB, of which PN is the measure (3 *Sph. Tr.*); therefore the rectangle under the sines of CA and CB, is to the square of radius, as the rectangle under the sines of the excesses of the half sum of all the sides AB, BC and AC above each of the legs AC and BC, is to the square of the sine of half the vertical angle ACB, and by alternating, the rectangle under the sines of AC and CB, is to the rectangle under the sines of the excesses of the half sum of AB, BC and AC above AC, and of the same half sum above BC, as the square of radius, is to the square of the sine of half the angle ACB (16. 5 *Eu.*); or, which is equal (20. 6 *Eu.*), in a duplicate ratio of radius to the sine of half the angle ACB.

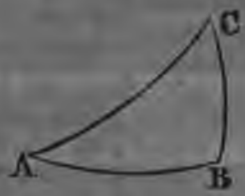
SOLUTIONS OF THE SEVERAL CASES OF SPHERICAL TRIGONOMETRY.

PROBLEM I.

Of the three sides and three angles of a right angled spherical triangle, any two being given, besides the right angle, to find the rest.

All the cases of this problem are solvible by prop. 23 and 24 Sph. Tr., for two of the five parts there mentioned being always given, and a third sought, these three parts must either be all contiguous, or one of them is not contiguous to either of the other two; in the first case, the part which is between the other two becomes the middle part, and the solution is made by prop. 24 and 16. 6 Eu.; in the other case, the part which is not contiguous to either of the other two becomes the middle part, and the solution is made by prop 23 and 16. 6 Eu. If a middle part be sought, you should begin with radius, if one of the extreme parts, with the other.

The solutions of the several cases of right angled spherical trigonometry are exhibited in the following table.



Case.	Part.	Given, besides the right angle.	Sought.	Solutions.
1.	1.	The hypothenuse and one leg AC and AB.	The other leg. BC.	By 23 Sph. Tr. and 16. 6 Eu., making the complement of AC the middle part, Cos. AB : R : . Cos. AC : Cos. BC. According as AC is less or greater than a quadrant, AB and BC are of the same or different affections, by 18 Sph. Tr.
	2.	The same.	The oblique angle adjacent to the given leg. A.	By 24 Sph. Tr. and 16. 6 Eu., making the complement of A middle part, R : T. AB : : Cot. AC : Cos. A. According as AC is less or greater than a quadrant, AB and BC, and therefore AB and A are of the same or different affections, by 18 Sph. Tr.

Case.	Part.	Given, besides the right angle.	Sought.	Solutions.
	3.	The same.	The oblique angle opposite the given leg. C.	By 23 Sph. Tr. and 16. 6 Eu., making AB the middle part, S. AC : R : : S. AB : S. C of the same affection as AB, by 16. Sph. Tr.
2.	1.	The hypothenuse and one oblique angle. AC and A.	The leg adjacent to the given obl. angle. AB.	By 24. Sph. Tr. and 16. 6 Eu., making the complement of A middle part, Cot. AC : R : : Cos. A : T. AB. According as AC is less or greater than a quadrant, A and C, and of course A and AB, are of the same or different affections, by 18 and 16 Sph. Tr.
	2.	The same.	The leg opp. the given obl. angle. BC.	By 23 Sph. Tr. and 16. 6 Eu., making BC the middle part, R : S. A : : S. AC : S. BC, of the same affection as A, by 16 Sph. Tr.
	3	The same.	The other obl. angle. C.	By 24 Sph. Tr. and 16. 6 Eu., making the comp. of AC middle part, Cot. A : R : : Cos. AC : Cot. C. According as AC is less or greater than a quadrant, A and C are of the same or different affections, by 18 Sph. Tr.

Case.	Part.	Given, besides the right angle.	Sought.	Solutions.
3.	1.	One leg and adj. obl. angle. AB and A.	The other obl. angle. C.	By 23 Sph. Tr. and 16. 6 Eu., making the comp. of C middle part, R : Cos. AB : : S. A : Cos. C, of the same affection as AB, by 16. Sph. Tr.
	2.	The same.	The other leg. BC.	By 24 Sph. Tr. and 16. 6 Eu., making AB middle part, Cot. A : R : : S. AB : T. BC, of the same affection as A, by 16 Sph. Tr.
	3.	The same.	The hypothenuse. AC.	By 24 Sph. Tr. and 16. 6 Eu., making the comp. of A middle part, T. AB : R : : Cos. A : Cot. AC.—According as AB and A are of the same or different affections, AC is less or greater than a quadrant, by 17 Sph. Tr.
4.	1.	One leg and the opp. angle. AB and C.	The other obl angle. A.	By 23 Sph. Tr. and 16. 6 Eu., making the comp. of C middle part, Cos. AB : R : : Cos. C : S. A, which is ambiguous.

Case.	Part.	Given, besides the right angle.	Sought.	Solutions.
4.	2.	The same.	The other leg. BC.	By 24 Sph. Tr. and 16. 6 Eu., making BC middle part, R : T. AB : : Cot. C : S. BC, ambiguous.
	3.	The same.	The hypothenuse AC.	By 23. Sph. Tr. and 16. 6 Eu., making AB middle part, S. C : R : : S. AB : S. AC, ambiguous.
5.	1.	Both legs. AB and BC.	The hypothenuse AC.	By 23 Sph. Tr. and 16. 6 Eu., making the comp. of AC middle part, R : Cos. AB : : Cos. BC : Cos. AC, which is less or greater than a quadrant, according as AB and BC are of the same or different affections, by 17 Sph. Tr.
	2.	The same.	Either obl. angle A.	By 24 Sph. Tr. and 16. 6 Eu., making AB middle part, T. BC : R : : S. AB : Cot. A, of the same affection, as BC, by 16 Sph. Tr.

Case.	Part.	Given, besides the right angle.	Sought.	Solutions.
6.	1.	Both obl. angles. A and C.	The hypothenuse AC.	By 24 Sph. Tr. and 16. 6 Eu., making the comp. of AC middle part, R : Cot, A : : Cot. C : Cos. AC, which is less or greater than a quadrant, according as A and C are of the same or different affections, by 17 Sph. Tr.
	2.	The same.	Either leg. AB.	By 23 Sph. Tr. and 16. 6 Eu., making the comp. of C middle part, S. A : Cos. C : : R : Cos. AB, of the same affection as the angle C, by 16 Sph. Tr.

PROBLEM II.

Of the three sides and three angles of an oblique angled spherical triangle, any three being given, to find the rest.

The cases of oblique spherical trigonometry, with their solutions, are as follow.

Fig. 1.

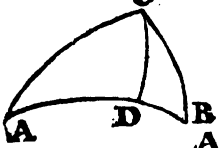
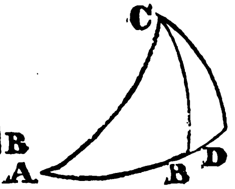


Fig 2.



Case.	Part.	Given.	Sought.	Solutions.
1.	1.	Two sides and an angle opposite one of them. AC, CB & A.	The remaining side. AB.	The perpendicular CD being, in both figures, let fall from C on AB; in the triangle ADC, right angled at D, AC and A being given, to find AD, by 24 Sph. Tr. taking the complement of A, as middle part, Cot. AC : R : : Cos. A : T. AD; and by Cor. 2. 26 Sph. Tr. Cos. AC : Cos. CB : : Cos. AD : Cos. DB; and AB is the sum or difference of AD and DB.
	2	The same.	The angle opposite the other given side. ABC.	By 27 Sph. Tr. S. BC : S. AC : : S. A : S. ABC.
	3.	The same.	The angle included by the given sides. ACB.	In the triangle ADC, right angled at D, AC and A being given, to find the angle ACD. By 24. Sph. Tr. taking the complement of AC as middle part, Cot. A : R : : Cos. AC : Cot. ACD, and by Cor. 4. 26 Sph. Tr. T. BC : T. AC : : Cos. ACD : Cos. BCD, and the angle ACB is equal to the sum or difference of the angles ACD and BCD.
2.	1.	Two sides & the included angle. AC, AB & A.	The other side. CB.	In the triangle ADC, right angled at D, AC and A being given, find AD, as in case 1 part 1 above, and by Cor. 2. 26 Sph. Tr. Cos. AD : Cos. DB : : Cos. AC : Cos. BC. According as BD and DC are of the same or different affections, BC is less or greater than a quadrant, by 17. Sph. Tr.

Case.	Part.	Given.	Sought.	Solutions.
2.	2.	The same.	An angle opposite one of the given sides. ABC.	In the triangle ADC, right angled at D, AC and A being given, find AD, as in case 1 part 1 above; and by Cor. 3. 26 Sph. Tr. S. DB : S. AD : : T. A : T. B. According as AB is greater or less than AD, the angles A and B are of the same or different affections (19. Sph. Tr.).
3.	1.	Two angles and a side opposite one of them. A, ABC & AC.	The other angle. ACB.	In the triangle ADC, right angled at D, AC and A being given, find the angle ACD as in case 1 part 3 above; and by Cor. 1. 26 Sph. Tr. Cos. A : Cos. B : : S. ACD : S. BCD ; and ACB is the sum or difference of ACD and BCD, according as the perpendicular CD falls within or without the triangle ACB, or (19. *Sph. Tr.*), according as the angle A and ABC are of the same or different affections.
	2.	The same.	The side between the given angles. AB.	In the triangle ADC, right angled at D, AC and A being given, find AD, as in case 1 part 1 above ; and by Cor. 3. 26 Sph. Tr. T. B : TA : : S. AD : S. DB ; and AB is the sum or difference of AD and DB.
	3.	The same.	The side opposite the other given angle. BC.	By 27. Sph. Tr. S. B : S. A : : S. AC S. BC.

Case.	Part.	Given.	Sought.	Solutions.
4.	1.	Two angles and the side between them. A, ACB & AC.	A side opp. one of the given angles. BC.	In the triangle ADC, right angled at D, AC and A being given, find the angle ACD, as in case 1 part 3 above; and by Cor. 4. 26 Sph. Tr. Cos. BCD : Cos. ACD : : T. AC : T. BC. If BCD and A, (and therefore, by 16. Sph. Tr. DB and DC), be of the same affection, BC is less than a quadrant; if BCD and A, and therefore DB and DC, be of different affections, BC is greater than a quadrant (17 *Sph. Tr*).
	2.	The same.	The other angle. ABC.	In the triangle ADC, right angled at D, AC and A being given, find the angle ACD, as in case 1 part 3 above; and by Cor. 1. 26 Sph. Tr. S. ACD : S. BCD : : Cos. A : Cos. B. According as the angle ACB is greater or less than ACD, the perpendicular CD falls within or without the triangle ACB, and therefore the angles A and ABC are of the same or different affections (19 *Sph. Tr*).

Case.	Part.	Given.	Sought.	Solutions.
5.		All the sides. AB, BC & AC.	Any angle. A.	From one of the angles not sought, as ACB, let fall the perpendicular CD on the opposite side AB ; let AC be the greater of the two sides AC and CB ; and if the perpendicular fall within the triangle, AB is the sum, if without, the difference, of the segments AD and DB ; in either case, find an arch, whose tangent is a fourth proportional to T $\frac{AB}{2}$, T. $\frac{AC+CB}{2}$ and T. $\frac{AC-CB}{2}$; if AB be the sum of AD and DB, the arch so found is half their difference, if AB be their difference, that arch is half their sum (28 *Sph. Tr.* and 16. 6 *Eu.*) ; in either case, the sum of that arch and the half of AB, is equal to the greater of the two AD and DB, and their difference, to the less (7 *Pl. Tr.*) ; and AD and DB being found ; in the triangle ADC, right angled at D, AD and AC being given, the angle A may be found by Prob. 1. solutions Sph. Tr. *Otherwise.* The rectangle under the sines of AB and AC : the rectangle under the sines of the arches $\frac{AB+BC+AC}{2}-AB$ and $\frac{AB+BC+AC}{2}-AC$: : the square of radius : the square of the sine of half the angle A (30 *Sph. Tr. and* 20. 6 *Eu*).

Fig. 1.

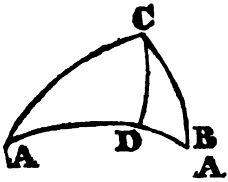
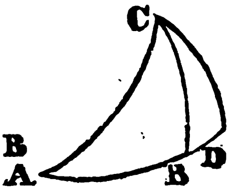


Fig. 2.



Case.	Part.	Given.	Sought.	Solutions.
6.		All the angles. CAB, ABC and ACB.	Any side. AC.	On one of the sides not sought, as AB, let fall a perpendicular CD from the opposite angle; let ABC be the greater of the two angles A and ABC; the perpendicular falls either within or without the triangle, according as the angles A and ABC are of the same or different affections (20 *Sph. Tr.*); in the former case, the angle ACB is the sum, in the latter, the difference, of the vertical angles ACD and BCD; in the former case, find an angle, whose tangent is a fourth proportional to Cot. $\frac{ACB}{2}$, T. $\frac{ABC+A}{2}$ and T. $\frac{ABC-A}{2}$; in the latter, an angle, whose cotangent is a fourth proportional to T. $\frac{ACB}{2}$, T. $\frac{ABC+A}{2}$ and T. $\frac{ABC-A}{2}$; the angle so found is, in the former case, half the difference, and in latter, half the sum of ACD and BCD (29 *Sph. Tr, and* 16. 6 *Eu.*); in either case, the sum of the angle so found and the half of ACB, is equal to the greater of the two ACD and BCD, and their difference to the less (7 *Pl. Tr.*); and ACD and BCD being found; in the triangle ADC, right angled at D, the angles A and ACD being given, AC may be found by prob. 1 solutions Sph. Tr.

Case.	Part.	Given.	Sought.	Solutions.
6.		All the angles. CAB, ABC and ACB.	Any side. AC.	*Otherwise.* Let DEF, see fig to prop. 13 Sph. Tr., be the supplemental triangle to the triangle ABC ; the arch DE is the complement of the angle ACB, EF of the angle BAC, and DF of the angle ABC, to semicircles ; the sides of the triangle DEF are therefore given ; from which, by case 5, find the angle DEF which is opposite the sought side AC ; which side may of course be found, being the complement of the measure of the angle DEF to a semicircle (13 *Sph. Tr*).

In the preceding solutions of the several cases of oblique angled spherical triangles, the rules are given for ascertaining the affections of the arches or angles sought, and removing ambiguities, where it could be conveniently done. For farther remarks on this subject, and particularly on the first solutious of the fifth and sixth cases, deduced from prop. 28 and 29 Sph. Tr. see note on Problem 2 Spherical Trigonometry.

ELEMENTS OF

NATURAL PHILOSOPHY,

As far as it relates to Astronomy, according to the Newtonian System.

Philosophy, which signifies a knowledge of things, is a word of Greek origin, and in that language means, a love of knowledge. It is divided into Moral and Natural. *Moral Philosophy,* which is also called *Etticks*, and by some *Metaphysicks,* treats of the duties and conduct of man, considered as a rational being. *Natural Philosophy*, called also *Physicks*, treats of the properties of natural things, the causes of the different phenomena or appearances, and the laws, by which the various operations, which we observe in natural things, are regulated; and of such natural laws, as may be applied to various useful purposes.

The assemblage of natural bodies or things, is called the *Universe.*

Though it is by no means the intention of this little tract to enter into the business of Natural Philosophy, farther than may be necessary to explain the motions of the heavenly bodies, and the laws by which these motions are regulated, deduced from the laws of motion; yet it seems not unimportant, previously to mention some of the principal axioms of philosophy, which have been deduced from common and constant experience; which are so evident, and so generally known, that a recital of a few of them will be sufficient.

1. *Nothing has no property.* Hence,

2. *No substance or being can be produced from nothing by any created being.*

3. *Matter cannot naturally be annihilated, or reduced to nothing; and though things may appear to be utterly destroyed, as, for instance, by the action of fire, by evaporation, &c., yet in such cases the substances are not annihilated, but they are only dispersed, or divided into particles, so minute as to elude our senses.*

4. *Every effect has, or is produced by, a cause, and is proportionate to it.*

The *rules* of reasoning in Philosophy, which have been formed after mature deliberation, are as follow:

Rule 1. That more causes of natural things ought not to be admitted, than are both true, and sufficient to explain their appearances.

Philosophers say, Nature does nothing in vain; and that is done in vain by more causes, which can be done by fewer.—— For nature is simple, and abounds not in superfluous causes of things.

Rule 2. Therefore of natural effects of the same kind, the same causes are, as far as possible, to be assigned.

As of respiration in a man, and in a beast; of the descent of stones in Europe and in America; of the light of a culinary fire and of the sun; of the reflection of light in the earth and in the planets.

Rule 3. The qualities of bodies which can neither be increased or diminished, and which are found in all bodies on which we can make experiments, are to be reputed qualities of all bodies whatever.

Such as the extension, hardiness, impenetrability, mobility and vis inertiæ of matter. And if it appear from experiments and astronomical observations, that all bodies about the earth gravitate towards the earth, and that, in proportion to the quantity of matter in each; that the moon, according to its quantity of matter, gravitates towards the earth, and our sea towards the moon; and all the planets and comets towards each other and the sun; we must by this rule affirm, that all bodies whatever gravitate towards each other. Indeed the argument from the appearances, for the universal gravitation of bodies, is stronger than for their impenetrability, of which we can have no experiment or observation in the celestial bodies.

Rule 4. In experimental philosophy, we should consider propositions collected by general induction from phenomena, as accurately or very nearly true, notwithstanding any contrary hypotheses which may be imagined, till other phenomena occur, by which they may be made more accurate, or liable to exceptions.

This rule should be followed, that the argument of induction may not be evaded by hypotheses.

These rules are evidently formed, in order that in our enquiries about the nature of bodies, we may be rather directed by experiment, than by hypotheses not founded on experiment, as appears to have been often done, to the evident danger of being led into errors; and as the object of research in these elements, is the system of the world, and to investigate the causes, from whence motions so accurate and beneficial are produced; it seems proper to mention previously, some of the principal laws of the planetary motions, discovered by that eminent astronomer, John Kepler, from actual observations, according to the Copernican hypothesis, among which are the following:

1st. The areas, which the planets, which revolve round the sun, describe by right lines drawn to it, are proportional to the times.

2nd. The orbits, which they describe, are not circles, as was before generally supposed, but ellipses, the sun being in one of the focuses.

3rd. The cubes of their mean distances from the sun are to each other, as the squares of their periodick times.

The two first laws being applicable to the moon's motion round the earth, and all three to the motion of Jupiter and Saturn's satellites round their primaries. It remained for the great Newton to deduce these and other laws of the system of the world, from the laws of motion, by mathematical reasoning. Some of his principal discoveries on this subject are delivered in the following elements.

DEFINITIONS.

1. *The quantity of matter*, is a measure thereof, arising from its density and magnitude jointly.

The air, for instance, its density being doubled, in a double space is four-fold, in a triple, six-fold. This quantity may be ascertained by its weight, especially in an exhausted receiver.

2. *The quantity of motion*, is a measure thereof, arising from the velocity and quantity of matter jointly.

The motion of the whole, is the sum of the motions of all the parts, and therefore in a body of double the quantity of matter, with an equal velocity, is double, and with a double velocity, four-fold. And ever so small a power may be made to move ever so great a weight; namely, by making the velocity of the power compared with that of the weight such, that the product of the quantity of matter of the power multiplied by its velocity, may be greater than the product of the quantity of matter of the weight by its velocity, and so much greater as to overcome such resistance as may arise from friction, &c.

3. *The force of inertness, or vis inertiæ, or vis insita of matter,* is the power of resisting, by which every body, as much as is in it, perseveres in its state of rest, or of uniform motion in a right line.

This force is proportional to the quantity of matter.

4. *An impressed force,* is an action exercised on a body, to change its state of rest, or uniform motion in a right line.

This force consists in the action alone, nor does it remain in the body after the action. For the body perseveres in every new state by its force of inertness alone. But the impressed force is of different origins, as from a stroke, a pressure, a centripetal or centrifugal force.

5. *A centripetal force,* is that, by which bodies are drawn, impelled, or any how tend towards any point as a centre.

Of this kind is *gravity,* by which bodies tend to the centre of the earth; magnetism, by which iron is attracted towards a magnet; and that force, whatever it be, by which the planets are perpetually drawn from rectilineal motions, and caused to be revolved in curve lines. A stone, whirled about in a sling, endeavours to recede from the hand which turns it; and by that endeavour, distends the sling, and with so much the greater force, as it is revolved with the greater velocity; and as soon as it is let go, flies away. That force which opposes itself to this endeavour, and by which the sling perpetually draws back the stone towards the hand, and retains it in its orbit, because it is directed towards the hand as the centre of the orbit, may be called the centripetal force. And the same thing is to be understood of all bodies revolved in any orbits. They all endeavour to recede from the centres of their orbits, and were it not for the opposition of a contrary force, by which they are retained in their orbits, and which may therefore be called centripetal, would go off in right lines with a uniform motion. A projectile, if it were not for the force of gravity, would not deviate towards the earth, but would go off in a right line, and with a uniform motion, if the resistance of the air were taken away. By its gravity it is perpetually drawn aside from its rectilineal course, and made to deviate towards the earth more or less, according to the force of its gravity, and the velocity of its motion.—By how much the less the force of gravity is, and the greater the velocity, with which it is projected, by so much the less it will deviate from a rectilineal course, and the farther it will go. If a leaden ball, projected from the top of a mountain, by the force of gun-powder, with a given velocity, in a horizontal direction, be carried to the distance of two miles before it falls to the ground; the same, with a double or ten-fold velocity, would go about double or ten times as far, provided the resistance of the air was taken away. And, by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line described by it, till at length it might describe ten, twenty, or ninety degrees, or even go round the whole earth, before it would fall; or finally might never fall, but might go off into the celestial spaces, and by the motion of going off, might proceed in infinitum. And in the same manner, as a projectile may, by the force of gravity, be made to revolve in an orbit, and go round the whole earth; the moon also may, either by the force of gravity, if it have gravity, or by any other force, by which it may be urged towards the earth, be perpetually drawn from a rectilineal course towards the earth, and made to revolve in its orbit: and without such a force the moon could not be retained in its orbit. If this force were too small, it would not sufficiently turn the moon from a rectilineal course; if too great, it would turn it too much, and draw it down from its orbit towards the earth. It is requisite, that the force be of a just quantity, and it belongs to mathematicians to find the force, by which a body may be accurately retained in any given orbit, with a given velocity; and again, to find the curvilineal path, into which a body going from a given place, with a given velocity, would be turned, by a given force.

Scholium.

When the word centripetal force, attraction, impulse or propension is used, the reader is to be aware, that, by these words, it is not meant, to determine the species or mode of action, or the physical cause or reason of it; or to ascribe these forces truly and physically to the centres, which are mathematical points, when the centres are said to attract, or forces are called centripetal; these forces being in this tract considered, not physically, but mathematically.

The terms, time, space, place and motion, are not explained in the above definitions, as being well known to all. But in order to avoid certain prejudices which may arise from the common conceptions of these things, it seems proper to distinguish them into absolute and relative, true and apparent, mathematical and common. In explaining the distinction between these, which appears extremely obvious, I shall be very brief.

1. Absolute, true, and mathematical time, in itself and its nature, flows equably, without relation to any thing external; and by another name is called, *duration:* Relative, apparent and common time, is some sensible and external measure of duration, by motion, whether accurate or inequable, commonly used for a true measure of time, as a day, a month, a year. Natural days are unequal, and astronomers correct the inequality, for the purpose of calculating the celestial motions. It is possible there may be no perfectly equable measure of time. All motions may be accelerated or retarded, but the flow of absolute time cannot be changed, and is the same, whether motions be swift or slow or none.

2. Absolute space, in its own nature, without relation to any thing external, remains always the same and immoveable. Relative space, is some moveable measure or dimension of this space, which is determined by our senses from its situation with respect to bodies, and is commonly reckoned immoveable: as the dimension of a subterraneous, aerial or celestial space determined by its situation with respect to the earth; and if the earth be moved, a space of our air, which, relatively and with respect to the earth, always remains the same, becomes at one time, one part of absolute space, and, at another time, another, and so its portion of absolute space is continually changed.

3. Place, is the part of space which a body occupies, and is, according to the nature of the space, either absolute or relative. Thus a body on this earth, which, apparently and with respect to the earth, remains in the same place, if the earth move, is continually changing its place or situation with respect to absolute or immoveable space; but situations, properly speaking, have not quantity, and are not so much places, as affections of places.

4. Absolute motion, is the translation of a body from an absolute place to an absolute place; Relative from a relative one to another. Thus if the earth move, a body on it may be relatively at rest, that is, with respect to the earth, and yet, with respect to absolute space, be in motion.

Hence it appears, that relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them, accurate or inaccurate, which are commonly used instead of the measured quantities themselves; and, if the meaning of words is to be determined by their use, by those names of time, space, place and motion, these measures are to be understood; and the language will be unusual, though purely mathematical, if the measured quantities themselves be meant.—

Therefore they do violence to the sacred writings, who there interpret these terms for the measured quantities themselves. Nor do they less contaminate mathematicks and philosophy, who confound the true quantities, with their relations and common measures.

To discover indeed the true motions of particular bodies, and actually to discriminate them, from those which are apparent, is a matter of no little difficulty; because the parts of that immoveable space, in which bodies are really moved do not strike our senses. Yet the cause is not entirely desperate. For arguments are within our reach, partly from the apparent motions, which are the differences of true ones, and partly from the forces, which are the causes of the true motions.

AXIOMS, OR LAWS OF MOTION.

Law 1. *That every body perseveres in its state of rest, or of moving uniformly in a right line, unless so far as it is compelled, by forces impressed thereon, to change that state.*

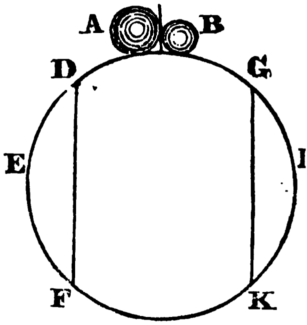
Projectiles persevere in their motions, unless so far as they are retarded by the resistance of the air, and impelled downward by the force of gravity. But the greater bodies of the planets and comets preserve their progressive and rotatory motions, in less resisting spaces, for a very long time.

Law 2. *That a change of motion is proportional to the force impressed, and according to the right line, in which that force is impressed.*

If any force generate a motion, a double force will generate a double motion, a triple force, a triple one, whether the force be impressed together and at once, or gradually and successively. And if the impressed body were previously in motion, and the impressed force be in the same direction, as that previous motion, the impressed motion is added to, or taken from, the previous motion, according as the two motions conspire with, or are contrary to each other. But if the impressed force be in an oblique direction, with respect to the previous motion, a new motion will arise compounded of the determination of both.

Law 3. *That reaction is always equal and contrary to action: or, that the actions of two bodies on each other are always equal, and directed to contrary parts.*

Whatever presses or draws another, is as much pressed or drawn by that other. If any one, with his finger press a stone, his finger is also pressed by the stone. If a horse draw a stone tied to a rope, the horse, if I may so speak, will be equally drawn back towards the stone: for the stretched rope, by the endeavour of relaxing itself, will urge the horse towards the stone, and the stone towards the horse, and will as much impede the progress of one, as it advances that of the other. If a globular body, as an ivory ball, impinging on another similar one, by its force change in any way the motion of that other, the same will also by the force of that other, on account of the equality of the mutual pressure, undergo an equal change in its motion, to the contrary part. By these actions, if the bodies be unequal, equal changes are made, not of velocities, but of motions; namely, in bodies not otherwise obstructed: for the changes of velocities, made towards contrary parts, because the motions are equally changed, are reciprocally proportional to the magnitudes of the bodies.



In attractions, which are the principal object of this tract, the truth of this law may be thus shewn. Between any two bodies A and B, mutually attracting each other, conceive any obstacle to be placed, by which their coming together may be hindered. If either body A be more attracted towards the other B, than that other B towards the former A, the obstacle would be more urged by the pressure of the body A, than by that of B, and therefore would not remain in an equilibrium. The stronger pressure would prevail, and cause, that the system of the two bodies and the obstacle would be moved directly towards the part, on which B lies, and in free spaces would go forward *in infinitum*, with a motion continually accelerated; which is absurd and contrary to the first law. For, by the first law, the system ought to persevere in its state of rest, or of moving uniformly forward in a right line; therefore the bodies must equally press the obstacle, and are equally attracted by each other. The truth of this may be shewn by experiment. in the attraction between a magnet and iron. If these, placed apart in proper vessels touching each other, float near each other in still water, neither will propel the other, but, by the equality of attraction both ways, they will sustain each other's pressure, and at length rest in an equilibrium.

So also the gravity between the earth and its parts is mutual. Let the earth EH be cut by any plain DF into two unequal parts DEF and DHF; their weights towards each other are mutually equal. For if the greater part DHF be, by another plane GK parallel to the former DF, cut into two parts DFKG and GHK, of which the exterior part GHK is equal to the less part first cut off DEF: it is manifest, that the middle part DFKG will, by its own weight, tend to neither of the extreme parts, but will, if I may so speak, be suspended, and rest in an equilibrium between both. But the extreme part GHK would press with all its weight on the middle part, and urge it towards the other extreme part DEF; therefore the force with which the sum of the parts GHK and DFKG tends towards the third part DEF is equal to the weight of the part GHK, or, to the weight of the third part DEF. Therefore the weights of the two parts DEF and DHF towards each other are equal, as was proposed to be proved. And unless these weights were equal, the whole earth, floating in a free ether, would yield to the greater weight, and, in going from it, would be carried off *in infinitum.*

Cor. 2. And much more, the rectilineal figure, which is comprehended under the chords of the evanescent arches ab, bc, cd, &c. coincides ultimately with the curvilineal figure.

Cor. 3. As also the circumscribed rectilineal figure, comprehended under the tangents of the same arches.

Cor. 4. And therefore these ultimate figures, (as to their perimeters acE,) are not rectilineal, but curvilineal limits of rectilineal figures.

LEMMA IV.

If in two figures there be inscribed, as in the preceding lemma, two ranks of parallelograms, an equal number in each figure, and, when their breadths are diminished in infinitum, the ultimate ratios of the parallelograms in one figure to those in the other, each to each, be the same; these two figures are to each other, in the same ratio.

For as the parallelograms in one figure are to those in the other, each to each, so is the sum of all the parallelograms in the former to the sum of all in the other (12. 5 *Eu.*), and so is the former figure to the other, the former figure being to the former sum, and the latter figure to the latter sum, in the ratio of equality (*by Lemma* 3).

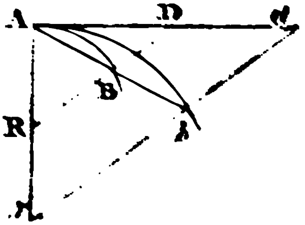
Cor. Hence if two quantities of any kind, be any how divided into an equal number of parts, and these parts, when their number is increased, and magnitude diminished in infinitum, have a given ratio to each other, the first to the first, the second to the second, and the others in their order to the others; the whole quantities are to each other in the same given ratio. For if, in two such figures, as those mentioned in this lemma, parallelograms be taken, which are to each other as the parts, the sum of the parts are always as the sum of the parallelograms (12 and 11. 5 *Eu.*), and therefore, when the number of the parts and parallelograms is increased and their magnitude diminished in infinitum, in the ultimate ratio of a parallelogram to its correspondent one, or which is equal (*Hyp.*), of one of the parts to its correspondent one.

LEMMA V.

All corresponding sides of similar figures, as well curvilinear, as rectilinear, are proportional: and the areas are in a duplicate ratio of the corresponding sides.

LEMMA VI.

If any arch (ACB), given by position, be subtended by a chord (AB), and, in any point (A) in the middle of the continued curvature, be touched by a right line (AD), produced both ways; then as its extreme points (A and B) approach each other and come together; the angle (BAD) contained by the chord and tangent will be diminished in infinitum, and ultimately vanish.



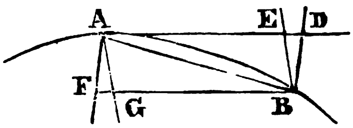
For if that angle should not vanish, the arch AB would contain, with the tangent AD, an angle equal to a rectilineal one, and therefore the curvature at the point A would not be continual, contrary to the supposition.

LEMMA VII.

The same things being supposed; the ultimate ratio of the arch, chord, and tangent to each other, is the ratio of equality.

For while the point B, see fig. to prec. lemma, approaches to the point A, let the right lines AB and AD be always understood to be produced to distant points b and d, and to the secant or cutting line BD, let bd be drawn parallel, and let the arch Ab be always similar to the arch AB. And, the points A and B coinciding, the angle dAb, by the preceding lemma, vanishes, and therefore the right lines Ab and Ad, which are always finite, and the intermediate arch Ab, coincide and are therefore equal. Whence also the right lines AB and AD, always proportional to the right lines Ab and Ad (4. 6 *Eu.*); and the intermediate arch AB, have to each other, an ultimate ratio of equality.

Cor 1. Whence if through B, a right line BF be drawn parallel to the tangent, always cutting any right line AF passing thro' A, in F; this right line BF, has ultimately to the evanescent arch AB, a ratio of equality; because, the parallelogram AFBD being completed, it has always a ratio of equality to AD.



Cor. 2. And if through B and A, there be drawn more right lines BE, BD, AG and AF, cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissas or right lines cut off AD, AE, BF, BG, and of the chord and arch AB, to each other, is the ratio of equality.

Cor. 3. And therefore all these lines, in all reasoning about ultimate ratios, may be used for each other.

LEMMA VIII.

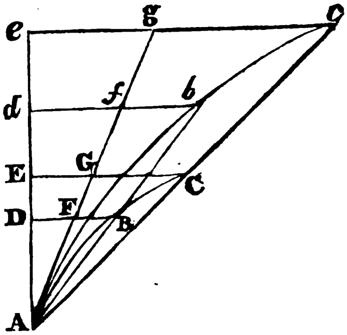
If the two right lines AR and BR, see the figure to lemma 6, with the arch AB, the chord AB, and the tangent AD, from three triangles ARB, ARB and ARD, and the points A and B approach and come together; the ultimate form of the evanescent triangles, is that of similitude, and the ultimate ratio, that of equality.

For while the point B approaches to the point A, let AB, AD and AR be always understood to be produced to distant points b, d and r, the right line rbd to be drawn parallel to RD, and let the arch Ab be always similar to the arch AB. And the points A and B coinciding, the angle dAb, by lemma 6, vanishes, and therefore the three triangles rAb, rAb and rAd, which are always finite, coincide, and are therefore similar and equal. Whence also the triangles RAB, RAB and RAD, which are always similar and proportional to these, become ultimately similar and equal to each other.

Cor. And hence, these triangles, in all reasoning about ultimate ratios, may be used for each other.

LEMMA IX.

If a right line AE, and a curve ABC, given by position, cut each other in a given angle at A, and to that right line, in another given angle, BD and CE be ordinately applied, meeting the curve in B and C, and the points B and C approach and come together to the point A; the areas of the triangles ABD and ACE are to each other ultimately, in a duplicate ratio of the sides.



For while the points B and C approach to the point A, let the right line AD be understood to be produced to distant points d and e, so that Ad and Ae may be proportional to AD and AE, and let the ordinates db and ec be drawn parallel to DB and EC, which may meet the right lines AB and AC produced in b and c. Let there be understood to be drawn, both the curve Abc similar to ABC, and the right line AG, which may touch both curves in A, and cut the ordinates DB, EC, db and ec in F, G, f and g. And, the length Ae remaining the same, let the points B and C come together to the point A, and, the angle cAg vanishing, the curvilineal areas Abd and Ace will coincide with the rectilineal ones Afd and Age, and therefore, by lemma 5, will be in a duplicate ratio of the sides Ad and Ae (19. 6 *Eu.*); but to these areas, the areas ABD and ACE, and to these sides, the sides AD and AE are always proportional; therefore the areas ABD and ACE are to each other ultimately in a duplicate ratio of the sides AD and AE.

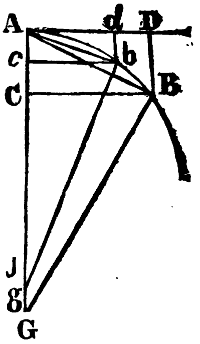
LEMMA X.

The spaces, which a body describes, by any finite force urging it, whether that force be determined and immutable, or be continually increased or continually diminished, are, in the very beginning of the motion, in a duplicate ratio of the times.

Let the times be represented by the right lines AD and AE, see figure to the preceding lemma, and the velocities generated, by the ordinates DB and EC; the spaces described by these velocities, are as the areas ABD and ACE described by these ordinates, or, which is equal by the preceding lemma, in a duplicate ratio of the times AD and AE.

LEMMA XI.

The evanescent subtense of the angle of contact, in all curves having a finite curvature at the point of contact, is ultimately in a duplicate ratio of the subtense of the conterminous arch.



Case 1. Let that arch be AB, its tangent AD, the subtense of the angle of contact BD, and the subtense or chord of the conterminous arch or the arch having the same extremes, the right line AB; and first, let the subtense BD of the angle of contact be perpendicular to the tangent AD. To the subtense AB and tangent AD, erect the perpendiculars BG and AG, meeting each other in G, and let the points D, B and G approach to the points d, b and g, and let j be the intersection of the right lines BG and AG, made ultimately, when the points D and B come to A. It is manifest that the distance GJ may be less than any given right line. But, because of the right angled triangle ABG, the square of AB is equal to the rectangle GAC (*Cor.* 1. 8 6 and 17. 6 *Eu.*), or, AC and BD being equal, to the rectangle under AG and BD; for the same reason, the square of the right line Ab is equal to the rectangle under Ag and bd; therefore the square of AB is to the square of Ab, in a ratio compounded of the ratios of AG to Ag and of BD to bd (23. 6 *Eu*). But because GJ may be assumed less than any given length, the ratio of AG to Ag may be such, as to differ from the ratio of equality less than by any given difference, and therefore the ratio of the square of AB to the square of Ab may be such, as to differ from the ratio of BD to bd less than by any given difference; and therefore, by lemma 1, the ultimate ratio of the square of AB to the square of Ab is equal to the ultimate ratio of bD to Bd, and so bD is to bd ultimately in a duplicate ratio of the subtenses AB and Ab (20. 6 *Eu*).

Case 2. Let now BD be inclined to AD in any given angle, and the ultimate ratio of BD to bd will still be the same as before (4. 6 *Eu.*), and therefore in a duplicate ratio of the subtenses AB and Ab.

Cnse 3. And though the angle D should not be given, but be formed by the right line BD converging to a given point, or by any other law; yet the angles D and d, constituted by a common law, always tend to equality, and approach nearer to each other, than by any given difference, and are therefore ultimately equal, by lemma 1; and therefore the lines BD and bd are to each other, in the same ratio as before.

Cor. 1. Since the tangents AD and Ad, the arches AB and Ab, and their sines BC and bc, become ultimately equal to the chords AB and Ab; their squares are ultimately, as the subtenses BD and bd.

Cor. 2. Their squares are also ultimately as the sagittas or versed sines of the arches, bisecting the chords, and tending to a given point. For these sagittas are as the subtenses BD and bd.

Cor. 3. And therefore the sagitta is in a duplicate ratio of the time, in which a body with a given velocity describes an arch; the arch described with a given velocity being as the time.

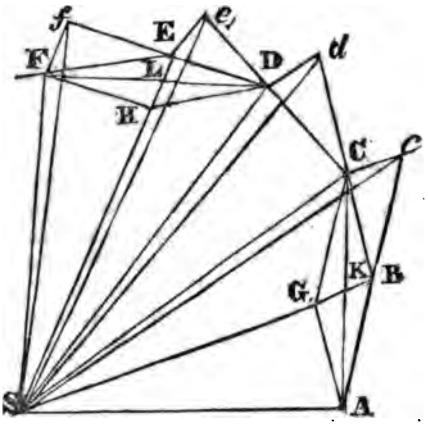
Cor. 4. The rectilineal triangles ADB and Adb are ultimately in a triplicate ratio of the sides AD and Ad, and in a sesquiplicate of the sides DB and db, as being in a ratio, compounded of the ratios of the sides AD and DB to Ad and db (*Cor.* 1. 23. 6 *Eu*). So also the rectilineal triangles ABC and Abc are to each other ultimately in a triplicate ratio of the sides BC and bc (*See Def.* 14. 5 *Eu*).

Cor. 5. And because DB and db are ultimately parallel, and in a duplicate ratio of AD and Ad, and therefore AC and Ac ultimately in a duplicate ratio of BC and bc, which is the nature of the parabola (*Cor.* 2. 40. 1 *Sup.*); the ultimate curvilineal areas ADB and Adb are two thirds parts of the rectilineal triangles ADB and Adb (*Cor.* 81. 1 *Sup.*); and therefore the ultimate segments AB and Ab third parts of the same triangles. And therefore these areas and segments are in a triplicate ratio both of the tangents AD and Ad, and of the chords of the arches AB and Ab.

In subsequent citations, Nat. Ph. denotes, Natural Philosophy.

PROP. I. THEOR.

The areas, which revolving bodies describe, by radiuses drawn to an immoveable centre of force, are in immoveable plains, and proportional to the times.



Let the time be divided into equal parts, and, in the first part of time, let the body describe, by its innate force, the right line AB. The same would, in the second part of time, if nothing hindred, go on directly to c, describing the right line Bc equal to AB (*by law* 1), so that the radiuses AS, BS and cS being drawn to the centre S, there would be described equal areas ASB and BSc (38. 1 *Eu.*): but, when the body comes to B, let a centripetal force act with a single but great impulse, and cause that the body deviate from the right line Bc, and go on in the right line BC. Through c draw cC parallel to BS, meeting BC in C; and, the second part of time being completed, the body (by cor. to the laws), will be found in C, in the same plain with the triangle ASB : join SC, and the triangle SBC, because of the parallels SB and Cc, is equal to the triangle SBc (37. 1 *Eu.*), and therefore to the triangle SAB. By a like reasoning if the centripetal force act successively in C, D, E, &c. causing that the body, in the several particles of time, describe the several right lines CD, DE, EF, &c. all these will be in the same plain, and the triangle SCD will be equal to the triangle SBC, the triangle SDE to SCD, and SEF to SDE; therefore equal areas are described in equal times in an immoveable plain, therefore any sums of the areas SADS and SAFS are to each other, as the times of their description (*Theor.* 2. 15. 5 *Eu*). Let now the number of these triangles be increased, and their breadth diminished in infinitum, and their ultimate perimeter ADF (*by cor.* 4 *lem.* 3), will be a curve line, as must be the case, since the centripetal force, by which the body is perpetually drawn from the tangent, is supposed to act unceasingly; and any described areas SADS and SAFS, which have been shewn to be always to each other, as the times of their description, are, in this case also, to each other, as the times of their description.

Cor. 1. The velocity of a body, attracted towards an immoveable centre, in non-resisting spaces, is inversely as the perpendicular let fall from that centre, on a rectilineal tangent of the orbit. For the velocity in the places A, B, C, D and E, are as the bases of equal triangles, namely, AB, BC, CD, DE and EF, and these bases are reciprocally as the perpendiculars let fall on them, as is manifest from 15. 6 Eu.

Cor. 2. If the chords AB and BC, of two arches successively described in equal times, in non-resisting spaces, by the same body, be completed into a parallelogram ABCG, and its diagonal BG, in that position, which it has ultimately when these arches are diminished in infinitum, be produced both ways; it will pass through the centre of force.

Cor. 3. If the chords AB and BC, DE and EF, of arches described in equal times in non-resisting spaces be completed into parallelograms ABCG and DEFH; the forces in B and E are to each other in the ultimate ratio of the diagonals BG and EH, when these arches are diminished in infinitum. For the motions of the body BC and EF (*by cor. to the laws*), are compounded of the motions Bc and BG, Ef and EH; and BG and EH, equal to Cc and Ff, in the demonstration of this proposition, were generated from the impulses of the centripetal force in B and E, and are therefore proportional to these impulses.

Cor. 4. The forces, with which, any bodies, in non-resisting spaces, are drawn from rectilineal motions, and turned into curvilineal orbits, are to each other, as those sagittas of arches, described in equal times, which tend to the centre of force, and bisect the chords, when these arches are diminished in infinitum. For the sagittas BK and EL, when these arches are so diminished, are halves of the diagonals, mentioned in the preceding corollary (*Schol.* 3. 13. 2 *Eu*).

Cor. 5. And therefore, the same forces, are to the force of gravity, as these sagittas, are to sagittas, perpendicular to the horizon, of the parabolick arches, which projectiles describe in the same time.

PROP. II. THEOR.

Every body, which is moved in any curve line described in a plain, and by a radius drawn to an immoveable point, describes areas about that point, proportional to the times, is urged by a centripetal force tending to the same point.

For every body, which is moved in a curve line, is turned from is rectilineal course, by some force acting on it (*by law* 1); and that force, by which a body is turned from a rectilineal course, and is made to describe the equal least possible triangles SAB, SBC, SCD, &c. see fig. to prec prop., about an immoveable point S, in equal times, acts, in the place B, according to a line parallel to cC (40. 1 *Eu.* and *Law* 2), or, according to the line BS; and, in the place C, according to a line parallel to dD, or, according to the line CS, &c. Therefore it always acts according to lines tending to that immoveable point S.

Cor. 1. In non-resisting spaces or mediums, if the areas be not proportional to the times, the forces do not tend to the concourse of the radius, but deviate therefrom, *in consequentia,* or towards the part to which the motion is directed, if the description of the areas be accelerated; but *in antecedentia,* if retarded.

Cor. 2. Even in resisting mediums, if the description of areas be accelerated, the directions of the forces deviate from the concourse of the radiuses, towards the part, to which the motion is made.

Scholium.—A body may be urged by a centripetal force compounded of several forces. In this case, the sense of the proposition is, that the force, which is compounded of all, tends to the point S. Moreover, if any force act according to a line perpendicular to the described surface, this will cause, that the body deviate from the plain of its motion, but will neither increase nor diminish the quantity of the described surface, and is therefore to be neglected in the composition of forces.

And since the equable description of areas is an *index* of the centre, which that force respects, by which a body is most affected, and by which it is drawn from a rectilineal motion, a retained in its orbit; *the equable description of areas,* is used in this tract, as the *index* of the centre, about which, all curvilineal motion is performed in free spaces.

PROP. III. THEOR.—*See Note.*

The centripetal forces, of bodies, which describe different circles with an equable motion, tend to the centres of the circles, and are to each other, as the squares of arches described together, applied to the radiuses of the circles.

These forces tend to the centres of the circles, by prop. 2 and cor. 2 prop. 1 Nat. Ph.; and are to each other, as the versed sines of the least possible arches, described in equal times (*Cor.* 4. 1 *Nat. Ph.*), or, which is equal (*Lem.* 7 *Nat. Ph.* 31. 3, *Cor.* 1 *to* 8. 6 *&* 17. 6 *Eu.*), as the squares of the same arches applied to the diameters of the circles; and therefore, since these arches, are as arches described in any equal times, and the diameters of circles, are as their radiuses, these forces are to each other, as the squares of any arches described together, applied to the radiuses of the circles.

Cor. 1. Therefore, since these arches, are as the velocities of the bodies, the centripetal forces are as the squares of the velocities, applied to the radiuses of the circles; or, in the language of geometers, in a ratio compounded of the duplicate ratio of the velocities and the inverse simple ratio of the radiuses.

Cor. 2. And, since the periodick times, are in a ratio compounded of the direct ratio of the radiuses and the inverse one of the velocities; the centripetal forces are inversely as the squares of the periodick times applied to the radiuses of the circles; that is, in a ratio, compounded of the direct ratio of the radiuses and the inverse duplicate one of the periodick times.

Cor. 3. Whence, if the periodick times be equal, and therefore the velocities be as the radiuses; the centripetal forces are as the radiuses: and the contrary.

Cor. 4. If the periodick times, and therefore the velocities, be in a subduplicate ratio of the radiuses; the centripetal forces are equal: and the contrary.

Cor. 5. If the periodick times be as the radiuses, and therefore the velocities equal; the centripetal forces are inversely as the radiuses: and the contrary.

Cor. 6. If the periodick times be in a sesquiplicate ratio of the radiuses, and therefore the velocities in an inverse subduplicate ratio of the radiuses; the centripetal forces are inversely as the squares of the radiuses: and the contrary.

Cor. 7. And universally, if the periodick time be as any power R^n of the radius R, and therefore the velocity inversely as R^{n-1}; the centripetal force is inversely as R^{2n-1}: and the contrary.

Cor. 8. All the same things, concerning the times, velocities, and forces, with which bodies describe similar parts of any similar figures, having their centres similarly posited in those figures, follow from the demonstration of this proposition and its corollaries, applied to these cases. And it is applied, by substituting the equable description of areas, for equable motion, and the distances of the bodies from the centres, for the radiuses.

Cor. 9. From the same demonstration, it follows also; that the arch, which a body, by revolving uniformly in a circle with a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the descent of the body performed in the same time by falling with the same given force.

Scholium. The case of the sixth corollary of this proposition, namely, that of the periodick times being in a sesquiplicate ratio of the distances, or, which is the same, of the squares of the periodick times being as the cubes of the distances, obtains in the planetary bodies, as has been observed by Kepler, see the third law discovered by him, mentioned in these elements of Natural Philosophy, in the preparatory observations; and therefore those things, which relate to a centripetal force, decreasing in a duplicate ratio of the distances from the centres, are more particularly explained in these elements.

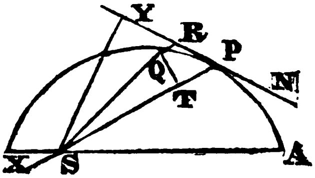
PROP. IV. THEOR.

If a body in a non-resisting space, be revolved in any orbit, about an immoveable centre, and describe any arch just nascent in the least possible time, and the sagitta be understood to be drawn, which may bisect the chord, and, being produced, may pass through the centre of force; the centripetal force in the middle of the arch, is as the sagitta directly, and the square of the time inversely.

For the sagitta in a given time is as the force (*Cor.* 4. 1 *Nat. Ph.*), and by increasing the time in any ratio, because the arch is increased in the same ratio, the sagitta is increased in a ratio which is duplicate of that ratio (*Cor.* 2 and 3 *Lem.* 11 *Nat. Ph.*), and therefore is as the force and square of the time jointly.—Taking from each the duplicate ratio of the time, the force is, as the sagitta directly, and the square of the time inversely.

The same may also be demonstrated from Lem. 10. Nat. Ph. thus :

The spaces, which a body describes, by any finite force urging it, whether that force be immutable or continually increased or diminished, are, in the very beginning of the motion, in a duplicate ratio of the times (*Lem.* 10. *Nat. Ph.*), and therefore, the forces being varied, as the forces and squares of the times jointly. Taking from each the duplicate ratio of the times, the forces are, as the spaces described directly, and the squares of the times inversely, and these spaces are as the sagittas mentioned in this proposition, as is manifest from Cor. 4. 1 Nat. Ph.



Cor. 1. If a body P, in revolving round a centre S, describe a curve line APQ and a right line RPN touch that curve in any point P, and from any other point of the curve Q, a right line QR be drawn parallel to the distance SP, and a perpendicular QT be drawn to that distance SP : the centripetal force is inversely as the solid $\frac{SP^2 \times QT^2}{QR}$, if that quantity of this solid be always taken, which it has ultimately when the points P and Q coincide.

For QR is equal to the sagitta of double the arch QP, in the middle of which is P; and double the triangle SQP, or SP×QT, is proportional to the time in which that double arch is described (1 *Nat. Ph.*), and therefore may be used, as an exponent of the time.

Cor. 2. By a similar reasoning, a perpendicular SY being let fall, from the centre of the force S, on a tangent of the orbit PR, the centripetal force is inversely as the solid $\frac{SY^2 \times QP^2}{QR}$, for the rectangles SY×QP and SP×QT are equal, being each equal to double the triangle SQP.

Cor. 3. If the orbit be a circle, or contains the least possible angle of contact with a circle, having the same curvature, and the same radius of curvature at the point of contact P, and if PX be the chord of this circle, drawn from the body through the centre of force; the centripetal force is inversely as the solid

$SY^2 \times PX$. For QP^2 is equal to the rectangle $PX \times QR$ (*Schol. Theor.* 5. 4 *Eu.*), and therefore PX is equal to $\frac{QP^2}{QR}$, and may be substituted for it, in expressing the quantity of the solid, mentioned in the preceding corollary.

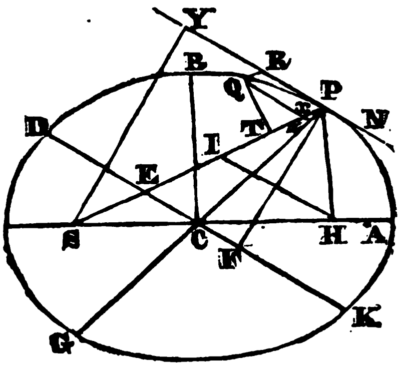
Cor. 4. The same things being supposed, the centripetal force is, as the square of the velocity directly, and that chord inversely. For the velocity is inversely as the perpendicular SY (*Cor.* 1. 1 *Nat. Ph*).

Cor. 5. Hence, if any curvilineal figure APQ be given, and in it a point S be also given, to which the centripetal force is continually directed; the law of the centripetal force may be found, by which, any body P, being continually drawn off from a rectilineal course, will be detained in the perimeter of that figure, and, in revolving, describe it. Namely, either the solid $\frac{SP^2 \times QT^2}{QR}$, or the solid $SY^2 \times PX$, should be computed, as inversely proportional to this force.

Scholium. Although the method of investigating centripetal forces, given in the preceding corollary, being the fifth of this proposition, is general, extending itself to any given curvilineal figure, and any point therein; yet as the principal object of these elements, is the investigation of those laws, which actually prevail in nature; and as Kepler has, from actual observation, ascertained, that the primary planets in their revolutions about the sun, describe ellipses, the sun being in one of the focuses; see the second law discovered by him, mentioned in these elements of natural philosophy, in the preparatory observations; and as the same law has been found, as far as observations have been made, to prevail in the motions of the secondary planets round their primaries. The investigation of the law producing a motion in an ellipse, round a focus, as the centre of force, is given in the next proposition; the like investigation, as respects a motion in a hyperbola or parabola, round a focus, as the centre of force, being given in the two following propositions.

PROP. V. PROB.

Let a body revolve in an ellipse; the law of the centripetal force, tending to its focus, is required.



Let S be the focus of the ellipse, to which the centripetal force tends, H the other focus, C the centre, CA and CB semi-axes, GP the diameter passing through the body P, DK the diameter conjugate thereto, Q a point in the perimeter APQ at the least possible distance from P, Qz an ordinate to the diameter GP, RPN a right line touching the ellipse in P; draw SP meeting DK in E and Qz in x, to the tangent RPN draw QR parallel to SP, on SP and DK let fall the perpendiculars QT and PF, and draw HI parallel to DK, meeting SP in I.

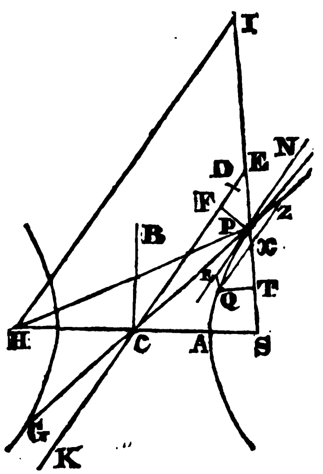
Because of the equals SC and CH, and the parallels EC and HI, SE is equal to EI (2. 6 *Eu.*), and because the angles IPR and HPN are equal (11. 1 *Sup.* and 15. 1 *Eu.*), and HI being parallel to RN (*Def.* 14. 1 *Sup.* and 30. 1 *Eu.*), and therefore the angles PIH and PHI equal to their alternates IPR and HPN (29. 1 *Eu.*), the angles PIH and PHI are equal, and therefore the right lines PH and PI (6. 1 *Eu.*); therefore EI is the half of SI, and IP of IP and PH together, and therefore EP is the half of SP and PH together, and therefore equal to the greater semiaxis CA (1. 1 *Sup*).

The principal parameter of the ellipse being called L; $L\times QR$ is to $L\times Pz$, as QR, or its equal (34. 1 *Eu.*), Px is to Pz (1. 6 *Eu.*), or, which is equal (2. 6 *Eu.*), as PE or AC is to PC; and $L\times Pz$ is to the rectangle GzP, as L is to Gz (1. 6 *Eu.*); and the rectangle GzP is to the square of Qz, as the square of CP is to the square of CD (40. 1 *Sup.*); and the ratio of the square of Qz to the square of Qx, the points Q and P coming together, is the ratio of equality (*Cor.* 2 *Lem.* 7 *Nat. Ph.*); and the triangles QxT and PEF being, because of the right angles at T and F, and the angles at x and E equal, being alternate angles (29. 1 *Eu.*), equiangular, the square of Qx, or of its equal

Qz is the square of QT, as the square of PE or AC is to the square of PF (4 and 22. 6 *Eu.*), or, which is equal (53. 1 *Sup.* 35. 1, and 16 and 22. 6 *Eu.*), as the square of CD is to the square of CB; and, compounding all these ratios, $L \times QR$ is to the square of QT, as $AC \times L \times PC^2 \times CD^2$, or, $AC \times L$ being equal to $2CB^2$ (*Def.* 15 and 17. 1 *Sup.* and 17. 6 *Eu.*), as $2CB \times PC^2 \times CD^2$ is to $PC \times Gz \times CD^2 \times CB^2$ (22. 5 *Eu.*), or, applying each to $CB^2 \times PC \times CD^2$, which is common to both, as $2PC$ is to Gz; but, the points Q and P coming together, $2PC$ and Gz are equal; therefore $L \times QR$ and QT^2, which are proportional to these, are equal (*Cor.* 13. 5 *Eu*). Let these equals be drawn into $\frac{SP^2}{QR}$, and $L \times SP^2$ is equal to $\frac{SP^2 \times QT^2}{QR}$, or (*Cor.* 1. 4 *Nat. Ph.*), inversely as the centripetal force; whence, L being a given quantity, the centripetal force is inversely as SP^2, or in an inverse duplicate ratio of the distance SP.

PROP. VI. PROB.

Let a body be moved in a hyperbola; the law of the centripetal force, tending to its focus, is required.



Let S be the focus of the hyperbola, to which the centripetal force tends, H the other focus, C the centre, CA and CB semiaxes, GP the diameter passing through the body P, KD the diameter conjugate thereto, Q a point in the perimeter AQP at the least possible distance from P, Qz an ordinate to the diameter GP, RPN a right line touching the hyperbola in P; draw SP meeting and QZ in x and KD in E, to the tangent RPN draw QR parallel to SP, on SP and KD let fall the perpendiculars QT and PF, and draw HI parallel to KD meeting SP produced in I.

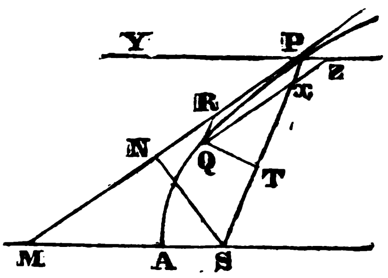
Because of the equals HC and CS, and the parallels CE and HI, SE is equal to EI (2. 6 *Eu.*), therefore PE is equal to half the difference of PI and PS, or, the angles HPR and IPN being equal (11. 1 *Sup.* and 15. 1 *Eu.*), and therefore, RN and HI being parallel (*Def.* 14. 1 *Sup.* and 30. 1 *Eu.*), their alternates PHI and PIH (29. 1 *Eu.*), and therefore the right lines PI and PH (6. 1 *Eu.*), PE is equal to half the difference of HP and PS, and therefore to the transverse semiaxis CA (1. 1 *Sup*).

The principal parameter of the hyperbola being called L; L×QR is to LxPz, as QR, or its equal (34. 1 *Eu.*), Px is to Pz (1. 6 *Eu.*), or, which is equal (2. 6 *Eu.*), as PE or AC is to PC; and LxPz is to the rectangle GzP, as L is to Gz (1. 6 *Eu.*); and the rectangle GzP is to the square of Qz, as the square of CP is to the square of CD (40. 1 *Sup.*); and the ratio of the square of Qz to the square of Qx, the points Q and P coming together, is the ratio of equality (*Cor.* 2 *Lem.* 7 *Nat. Ph.*); and the triangles QxT and PEF being, because of the right angles at T and F, and the angles at x and E equal, the external to the internal remote on the same side (29. 1 *Eu.*), equiangular, the square of Qx, or of its equal Qz is to the square of QT, as the square of PE or AC is to the square of PF (4 and 22. 6 *Eu.*), or, which is equal (53. 1 *Sup.* 35. 1, and 16 and 22. 6 *Eu.*), as the square of CD is to the square of CB; and, compounding all these ratios, LxQR is to the square of QT, as

$AC \times L \times PC^2 \times CD^2$, or, $AC \times L$ being equal to $2CB^2$ (*Def.* 15 and 17. 1 *Sup.* and 17. 6 *Eu.*), as $2CB^2 \times PC^2 \times CD^2$ is to $PC \times Gz \times CD^2 \times CB^2$ (22. 5 *Eu.*), or, applying each to $CB^2 \times PC \times CD^2$, which is common to both, as $2PC$ is to Gz; but the points Q and P coming together, $2PC$ and Gz are equal; therefore $L \times QR$ and QT^2, which are proportional to them, are equal (*Cor.* 13. 5 *Eu*). Let these equals be drawn into $\frac{SP^2}{QR}$, and $L \times SP^2$ is equal to $\frac{SP^2 \times QT^2}{QR}$, or (*Cor.* 1. 4 *Nat. Ph.*), inversely as the centripetal force; whence, L being a given quantity, the centripetal force is inversely as SP^2, or in an inverse duplicate ratio of the distance SP.

PROP. VII. PROB.

Let a body be moved in a parabola; the law of the centripetal force, tending to its focus, is required.



Let AQP be the parabola, S its focus, A the principal vertex, Yz the diameter passing through the body P, Q a point in the perimeter AQP at the least possible distance from P, Qz an ordinate to the diameter Yz, MP a right line touching the parabola in P, and meeting the axis in M; join SP meeting Qz in x, draw QR to the tangent MP parallel to SP, and on SP and MP let fall the perpendiculars QT and SN.

Because the angle SPM is equal to the angle YPM (11. 1 *Sup.*), or, which is equal (29. 1 *Eu.*), SMP, which is alternate to it, SP and SM are equal (6. 1 *Eu.*), whence, the triangles Pxz and SPM being equiangular, Px, or its equal (34. 1 *Eu.*), QR, is equal to Pz; but the square of Qz is equal to the rectangle under Pz and the parameter of the diameter Yz (40. 1 *Sup.*), or, that parameter being equal to 4PS (*Def.* 16. 1 *Sup.*), to the rectangle under Pz and 4PS, or to that under QR and 4PS; but the points P and Q coming together, the ratio Qz to Qx is the ratio of equality (*Cor.* 2 *Lem.* 7 *Nat. Ph.*); therefore the square of Qx is, in that case, equal to the rectangle under QR and 4PS. And since the triangles QxT and SPN, having the angles at T and N right, and the angles QxT and SPN equal, the external to the internal remote on the same side (29. 1 *Eu.*), are equiangular, the square of Qx is to the square of QT, as the square of PS is to the square of SN (4 and 22. 6 *Eu.*), or which is equal (62. 1 *Sup.* and *Cor.* 2. 20. 6 *Eu.*), as PS is to SA, or, which is equal (1. 6 *Eu.*), as 4PSxQR is to 4SAxQR; but the square of Qx is above shewn to be equal to 4PSxQR, therefore the square of QT is equal to 4SAxQR (14. 5 *Eu.*); let these equals be drawn into $\frac{SP^2}{QR}$, and $\frac{SP^2 \times QT^2}{QR}$ is equal to $SP^2 \times 4SA$, and therefore, by Cor. 1. 4 Nat. Ph., the centripetal force is inversely as $SP^2 \times 4SA$, or, 4SA being a given quantity, the centripetal force is inversely as SP, or in an inverse duplicate ratio of the distance SP.

Cor. 1. From this and the two preceding propositions it follows, that if any body P, go from a place P, in the direction of any right line PR, with any velocity, and be at the same time urged by a centripetal force, which is inversely proportional to the square of the distance of the places from the centre; this body will be moved in one of the conick sections, having a focus in the centre of force; and the contrary. For a focus, the point of contact and position of the tangent being given, a conick section may be described, having a given curvature at that point. But the curvature is given, from the given velocity and centripetal force; and two orbits touching each other, cannot be described by the same centripetal force and the same velocity.

Cor. 2. If the velocity, with which a body goes from its place P, be that, by which, in any least possible particle of time, the lineola PR may be described, and the centripetal force be such, as in the same time to move the same body through the space QR; this body will be moved in some conick section, whose principal parameter is that quantity $\frac{QT^2}{QR}$, which is made ultimately, when the lineolas PR and QR are diminished in infinitum. In these corollaries, the circle is referred to the ellipse, and the case excepted, when a body descends in a right line to the centre.

PROP. VIII. THEOR.

If many bodies be revolved about a common centre, and the centripetal force be in a reciprocal duplicate ratio of the distance of the places from the centre; the principal parameters of the orbits, are in a duplicate ratio of the areas, which bodies, by radiuses drawn to the centre, describe in the same time.

For, by Cor. 2. 7 Nat. Ph. the principal parameter is equal to the quantity $\frac{QT^2}{QR}$, which is made ultimately, when the points P and Q come together; but the very little line QR, in a given time is as the generating centripetal force, or, which is equal (*Hyp.*), reciprocally as SP^2; therefore $\frac{QT^2}{QR}$ is as $QT^2 x SP^2$; therefore the principal parameter is in a duplicate ratio of the area QTxSP, and therefore in a duplicate ratio of the areas described in the same time.

Cor. Hence, the whole area of an ellipse, and that which is proportional to it (*Cor* 1. 78. 1 *Sup.*), the rectangle under its axes, is in a ratio, compounded of the subduplicate ratio of the principal parameter, and the ratio of the periodick time. For the whole area is as the area QTxSP drawn into the periodick time.

PROP. IX.

The same things being supposed; the periodick times in ellipses, are in a sesquiplicate ratio of the greater axes.

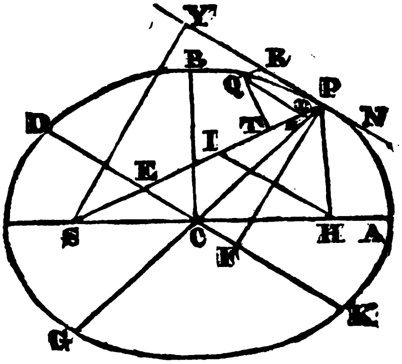
For the less axis is a mean proportional between the greater axis and the principal parameter (*Def.* 15 *and* 17. 1 *Sup.*), and therefore the square of the less axis is equal to the rectangle under the greater axis and that parameter (17. 6 *Eu.*), and therefore the less axis is in a subduplicate ratio of that rectangle; let there be added on each side, the ratio of the greater axis, and the rectangle under the axes is in a ratio, compounded of the subduplicate ratio of that parameter, and the susquiplicate of the greater axis ; but this rectangle (*by Cor.* 8 *Nat. Ph.*), is in a ratio, compounded of the subduplicate ratio of the principal parameter, and the ratio of the periodick time ; therefore the ratio compounded of the subduplicate ratio of the principal parameter, and the sesquiplicate of the greater axis, is equal to that which is compounded of the subduplicate ratio of the same parameter, and the ratio of the periodick time ; let there be taken away from each the subduplicate ratio of the principal parameter, and there remains the sesquiplicate ratio of the greater axis, equal to the ratio of the periodick time.

Cor. 1. Therefore the periodick times in ellipses, are the same, as in circles, whose diameters are equal to the greater axes of the ellipses.

Cor. 2. And the periodick times in ellipses, are in a sesquiplicate ratio of the mean distances of the revolving bodies from the centre of motion ; for these mean distances are equal to the greater semiaxes.

PROP. X. THEOR.

The same things being supposed, and right lines being drawn, touching the orbits at the bodies, and perpendiculars being let fall from the common focus on these tangents ; the velocities of the bodies are in a ratio, compounded of the inverse ratio of the perpendiculars, and the direct subduplicate ratio of the principal parameters.



From the focus S, to tangent PR, let fall the perpendicular SY.

The velocity of the body P, is as the least possible arch PQ described in a given particle of time, or which is equal (*Lem.* 7 *Nat. Ph.*), as the tangent PR, or which is equal (because of the proportionals PR to QT and SP to SY by 34. 1 and 4. 6 Eu., as $\frac{SP\times QT}{SY}$, or, as SY reciprocally and SP×QT directly, and SP×QT is as the area described in a given time, or, which is equal (8 *Nat. Ph.*), in a subduplicate ratio of the principal parameter.

Cor. 1. The principal parameters, are in a ratio, compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

Cor. 2. The velocities of bodies, in their greatest and least distances from the common focus, are in a ratio, compounded of the inverse ratio of the distances, and the direct subduplicate ratio of the principal parameters. For the perpendiculars are then the distances.

Co. 3. And therefore the velocity in a conick section, in the greatest or least distance from the focus, is to the velocity in a circle at the same distance from the centre, in a subduplicate ratio of the principal parameter of the section to double that distance. For that double is the principal parameter of the circle.

Cor. 4. The velocities of bodies revolving in ellipses, in their mean distances from the common focus, are the same, as of bodies revolving in circles at the same distances; or, (*by Cor.* 6. 3 *Nat. Ph.*), in an inverse subduplicate ratio of the distances. For the perpendiculars then are the less semiaxes; and these are as mean proportionals between the distances, which then become equal to the greater semiaxes, and the principal parameters. Let this ratio inversely, be compounded with the subduplicate ratio of the principal parameters directly, and it becomes the subduplicate ratio of the distances inversely.

Cor. 5. In the same figure, or even in different figures, whose principal parameters are equal, the velocity of a body is inversely as the perpendicular let fall from the focus on the tangent.

Cor. 6. In a parabola, the velocity is in an inverse subduplicate ratio of the distance of the body from the focus of the figure; in the ellipse, it is more varied, in the hyperbola, less, than in this ratio. For the perpendicular let fall, from the focus of a parabola, on a tangent, is in a subduplicate ratio of the distance (*Cor.* 1. 63. 1 *Sup. and Co* 1. 20. 6 *Eu*). In the ellipse, the perpendicular is more varied, in the hyperbola, less (*Cor.* 2. 63. 1 *Sup*).

Cor. 7. In a parabola, the velocity of a body at any distance from the focus, is to the velocity of a body revolving in a circle at the same distance from the centre, in a subduplicate ratio of the number two to unity; in an ellipse, it is less, in a hyperbola, greater, than in this ratio; for, by Cor. 2 of this proposition and Def. 15. and 17. 1 Sup., the velocity in the vertex of the parabola is in this ratio, and by Cor. 6 of this, and Cor. 6. 3 Nat. Ph., the same ratio is kept in all distances. Hence also, since, in circles, under the law of the centripetal force here supposed, the velocities are in an inverse subduplicate ratio of the distances (*Cor* 6. 3 *Nat. Ph.*), the velocity in a parabola, is every where equal to that of a body revolving in a circle at half the distance; in an ellipse, it is less; in a hyperbola greater.

Cor. 8. The velocity of a body revolving in any conick section, is to the velocity of a body revolving in a circle at the distance of half the principal parameter of the section, as that distance, is to the perpendicular let fall from the focus on the tangent of the section. For the diameter of a circle being equal to its principal parameter (*Def.* 17. 1 *Sup.*), the principal parameters of the section and circle are equal, and therefore, by Cor. 5 of this proposition, the velocity in the section, is to the velocity in the circle, as the distance in the circle, which is the perpendicular on the tangent, is to the perpendicular on the tangent of the section.

Cor. 9. Whence, since the velocity of a body revolving in this circle, is to the velocity of a body revolving in a circle at any other distance, in an inverse subduplicate ratio of the distances (*Cor.* 6. 3 *Nat. Ph.*); by equality, the velocity of a body revolving in a conick section, is to the velocity of a body revolving in a circle at the same distance, as a mean proportional between that common distance and half the principal parameter of the section, is to a perpendicular let fall from the common focus on the tangent.

ELEMENTS OF ASTRONOMY.

The principles delivered in the preceding elements of natural philosophy, may be considered as rather mathematical than philosophical; principles, on which reasonings may be founded, and conclusions deduced in philosophical enquiries. It remains, that from these principles be taught the system of the world, and the elementary principles of astronomy. In order to render this subject more clear and satisfactory, the principal and most important parts of it are thrown into the form of propositions, in a mathematical manner.

And that the conclusions deduced may be founded on experiment and actual observation, and not on hypotheses formed arbitrarily; it appears necessary before propositions are introduced on this subject, to lay before the reader a general view of the system of the world, and of those luminous bodies which are continually offered to our attention, usually called, heavenly bodies, according to the decision of the most able astronomers and philosophers, founded on the most accurate observations and reasonings.

OF THE SYSTEM OF THE WORLD.

In treating on this subject, the first thing, which naturally arrests our attention, is this earth which we inhabit; of the size and shape of which we can have little doubt; it having been repeatedly sailed round, its shadow being often exhibited to us in eclipses of the moon, and the shape of its surface such, that it is well known, that the situation of any part of it may be determined, with great accuracy, by observations of the heavenly bodies; its shape has been found to be nearly that of a globe or sphere of about 7900, nearly 8000 miles diameter. The causes of various appearances found to take place in it, as the vicissitudes of day and night, diversity of seasons and other phenomena, are deferred, till I shall treat of its motions.

The next thing, which attracts our attention, is that assemblage of heavenly bodies, which, by their splendour, number and variety, so much adorn the expanse around us ; and here, who can sufficiently admire and revere that infinite wisdom, power and goodness, which is so manifest in the works of creation, as far as human intellect and observation can trace them ; it is both our duty and privilege to make his works, the subject of our enquiries and meditations, and the more we shall enquire into and meditate on them, the greater cause we shall find to love, admire and worship their Almighty Author.

Among the heavenly bodies, the sun is by far most remarkable, resplendent and interesting to us, being the chief source of heat and light to this earth ; the moon is the next most striking object among them, being a great source of light to us, in the absence of the sun, and of about the same apparent magnitude, as that luminary ; besides these, the visible heavens is every where crowded with a vast number of luminous bodies, of small, but unequal apparent magnitudes, of which, by far the greater number retain, and have retained, since observations have begun to made on them, apparently the same situation with respect to each other, and are therefore called *fixed stars* ; the few which are continually changing their situation among them, being called *planets*, a Greek word, which signifies wanderers.

These planets have been uniformly determined by astronomers to belong to that, which is called the *solar system*, a name derived from the sun, which is supposed to be at rest in the centre, while the planets called primary, with their moons or satellites, called also secondary planets, revolve round him at various distances. By spots on its disk, however, the sun has been found to revolve on its axis in 25 days, 6 hours. It is likewise supposed to have a small motion about the centre of gravity of the whole system, which common centre of gravity, it has been determined by calculation, would not, on account of the superior magnitude of the sun, even if all the planets were on the same side of it, deviate father from the centre of the sun, than about the length of its diameter ; which is supposed to be about 890,000 miles,

The most noted primary planets which revolve about the sun, and the only ones known to the ancients, are in number, six, of which the earth is one ; of the others, two are nearer to the sun, and three more remote from it, than the earth ; the nearest to the sun being Mercury, the next Venus, then the earth, the three others in order being Mars, Jupiter and Saturn. Modern discoveries have added a few more to the number, of which mention will be made in order; the figures, which these primary planets describe, in revolving round the sun, have been found by observation to be ellipses, the sun being in one of the focuses.

In the following table are exhibited their magnitudes, mean distances from the sun, as determined by observations of the transit of Venus, which is by far the most accurate mode, known, their periodick times, and the other most important parculars respecting them.

A TABLE of the diameters, mean distances, periodick times, &c. of the principal primary planets in the solar system.

Names of the planets.	Diameters in English miles.	Mean distances from the sun.	Periodick times.	Eccentriciety of the orbit, the mean distance being 1000.	Diurnal rotation.	Inclination of axis to their orbit.	Inclination of their orbits to the eclipticK.
			d. h. " '		h. '	° '	° '
Mercury	3,200	36,841,468	87 23 15 53	205.89	unknown	unknown	6 54
Venus	7,906	68,891,486	224 16 49 24	7.15	23 22	75 0	3 24
Earth	7,970	95,173,000	365 6 9 30	17.00	23 56	23 28	0 0
Mars	4,400	145,014,148	686 23 27 30	92.54	24 39	28 42	1 52
Jupiter	89,000	494,990,976	4,332 12 20 35	48.16	9 56	0 0	1 20
Saturn	79,000	907,956,130	10,759 6 36 26	57.35	10 16	0° 0' nearly	2 30

The other primary planets, are Ceres Ferdinandea, Pallas, Juno, Vesta, the Georgium Sidus or Herschel, and Hercules, the four first move in orbits between those of Mars and Jupiter, of which the first was discovered by M. Piazzi, 1st January, 1801; Pallas, by Dr. Olbers, 28th March 1802; Juno, by M. Harding, at Lilienthal, in Germany, 1st September, 1804, and Vesta, by Dr. Olbers, in March, 1807. It is remarkable of the two former, that their orbits cross each other; Pallas, coming nearer to the sun than Ceres in the perihelion, or nearer part of their orbits, but removing to a greater distance in their aphelion, or more remote part; which is occasioned by the great eccentricity of the orbit of Pallas, compared with that of Ceres. The magnitudes of these two planets have been variously stated by astronomers. Dr. Herschel computes the diameter of Ceres to be a hundred and sixty-two, and that of Pallas, ninety-five miles. The four planets have been considered by some astronomers, as of a different species from the other planets, and have obtained the appellation of Asteroids. The periodick time of Ceres is 4y. 7m. 10n.; of Pallas, 4y. 7m. 11d.; of Juno, 5y. 182d.; and of Vesta 3y. 182d.

The **Georgium Sidus**, discovered by Dr. Herschel, and so named in honour of George III, king of England, but generally called Herschel, after the discoverer, may be seen with almost any telescope; its distance from the sun is computed at eighteen hundred millions of miles; its periodick time is about eighty-two years; and it is supposed to be ninety-three times the size of the earth. Six moons have already been discovered to move round it, which require very powerful telescopes to discern them, and its remoteness from the sun renders it probable, that it has a still greater number.

Hercules, lately discovered by Dr. Olbers, is about three times the size of Jupiter, and performs its revolution round the sun, in about two hundred and eleven years, its distance from that luminary being computed to be three thousand and forty-seven millions of miles. It appears to the naked eye, like a star of the sixth magnitude, and is attended by seven moons, one of which is supposed to be twice as large as the earth.

Besides these primary planets and the moons which have been mentioned, the earth is attended in its revolution round the sun by one satellite or moon, Jupiter by four, and Saturn by seven. Mercury and Venus, viewed though a telescope, exhibit phases like the moon, which shews, that they shine only by a borrowed light, namely, the reflected light of the sun, as all the planets both primary and secondary, which revolve round the luminary, are supposed to do; two white circles have been discovered about the poles of the planet Mars, which are supposed by Dr. Herschel, to originate from the snow lying about these parts; Jupiter is remarkable for his belts, and Saturn for his ring.

The circle in the heavens, in the plain of which the earth moves round the sun, and in which of course the sun appears to move, as seen from the earth, is called the *Ecliptick;* the angles which the orbits of the other principal planets have been found to make with it, are exhibited in the above table; they all perform their motions round the sun from west to east, as does also the earth, which is therefore said, to be according to the order of the signs; the whole circle of the ecliptick, through which the sun performs its apparent annual motion, with a space of eight degrees on each side of it, within which all the planetary motions were, by the ancients, thought to have been performed, is called the *Zodiack,* and has been divided by astronomers into twelve equal parts called *Signs,* each sign containing thirty degrees.

The earth, besides its annual motion round the sun, has a motion from west to east on its axis, in the space of about a day, which causes an apparent diurnal motion of the sun, moon and other heavenly bodies in the contrary direction, or from east to west; it appears from the above table, that the other planets, there mentioned have, as far as has been ascertained by observation, diurnal rotations on their axes. But the proximity of Mercurity to the Sun and its consequent brilliancy, has hitherto prevented astronomers from determining the time of its rotation on its axis, or the position of that axis. Yet Mr. Shroeter is induced, from some observations, to believe, that it revolves on its axis in 24h 5′ 8″.

Of the Satellites or Moons by which the primary planets are attended, the most remarkable to us is our Moon, which attends the earth in her revolution round the sun, and revolves about the earth from any one particular point in the heavens or fixed star, to the same point or star again, in twenty-seven days, seven hours, forty-three minutes; from change to change, in twenty-nine days twelve hours, forty-four minutes, and a little more than three seconds. Her diameter is about two thousand, one hundred and eighty miles, and her mean distance from the centre of the earth, about two hundred and forty thousand miles. Her orbit makes an angle with the plain of the ecliptick of about 5° 18′, the mean eccentricity contains about fifty-five parts, of which the mean distance contains a thousand, but varies from about forty-four to sixty-six of such parts, according to the different positions of the sun and earth. She shines with the borrowed light of the sun, which causes her, according to her situation with respect to the sun, to appear to us full, gibbous or horned; when the moon, being opposite to the sun, gets within the earth's shadow, so that, on account of the interposition of the earth between her and the sun, the sun cannot shine on her, she becomes opaque, and is said to be eclipsed, totally or partially, according as the sun, by the interposition of the earth, is prevented from shining on the whole or a part of the surface turned to us; again, when at the time of the change, the moon gets so between the sun and us, as to prevent the whole or a part of the surface of the sun which is turned to us, from shining on us, the sun is said to be eclipsed, totally or partially, according as the whole or a part of that surface is so obscured; and astronomers suppose the diameter of the eclipsed body to be divided into twelve equal parts, called digits; and the magnitude of the eclipse is estimated by the number of these digits eclipsed at the moment of the greatest obscuration.

From what has been said, it is manifest, that, if the plain of the moon's orbit coincided with the plain of the ecliptick, there would be eclipses at every change and full of the moon; but as the plain of the moon's orbit makes an angle with the plain of the ecliptick of about 5° 18′, as is above observed, no eclipse will happen, unless at the time of the change or full, the moon be so near to a node or intersection of the plains of the orbits, as in the former case, to get between the sun and some part of the earth, and in the latter, to get within the earth's shadow; the limit within which eclipses can happen being in the former case about seventeen degrees, and in the latter about twelve from such an intersection.

The earth has, besides the annual and diurnal motion just mentioned, a very slow retrograde motion of its axis, about the pole of the ecliptick, of 50″ each year, or of one degree in 72 years; its whole revolution would therefore require 25920 years, its half revolution 12960, and a fourth part of it 6480 years. Which motion is to be ascribed, as is hereafter shewn in this work, to the spheroidal form of the earth.

This motion has a considerable effect, in regulating the proportional time of the sun's remaining on the different sides of the equator, during the earth's annual motion; the earth was in its aphelion, when the sun was in the first of cancer, or at the time of the longest day with the inhabitants of northern latitudes, about A. D. 1148, and since that, in the lapse of nearly 700 years, has varied from that situation, but a little more than nine degrees; the consequence is, that the period from the March to the September equinox, is about eight days longer than the residue of the year, and the inhabitants of northern latitudes have their summers so much longer, and winters so much shorter, than those of southern latitudes; to which appears chiefly to be ascribed, the superior degree of cold experienced in southern latitudes, compared with northern of the same distance from the equator, for the situation of land and water in southern latitudes seems more favourable to temperature. If the present order of things were to last till A. D. 7628, the earth's aphelion would take place, when the sun would be in the first of aries, and each side of the equator would have him an equal portion of the year; and if the same order were to continue 6480 years longer, that aphelion would happen, when the sun would be in the first of Capricorn, and the inhabitants of southern latitudes would have their summers eight days longer, and winters as much shorter, than those of northern latitudes.

From this motion arises also the phenomenon of the precession of the equinoxes, whereby the point among the fixed stars, in which the sun crosses the equinox in the 1st of aries, has a slow retrograde motion, or from east to west, of about one degree in 72 years, so that the point where the sun at this time crosses the equinox in March, is about a whole sign to the westward of the constellation of aries, near which it crossed it in the time of Hipparchus, who flourished about 150 years before the Christian era, and now crosses it in the constellation of pisces.

Besides the planetary bodies just mentioned, there belong to the solar system, other bodies, called *Comets*, which appear from time to time, and are chiefly distinguishable by tails which continually issue from them. They have been determined by astronomers, to be opaque bodies, receiving all their light from the sun, and to move round the sun in very eccentrick or oblong elliptical orbits; so eccentrick and so nearly approaching to the figure of Parabolas, that, while within our view, their observed places hardly differ sensibly from those arising from calculations founded on this hypothesis; their apparent magnitudes are very different, sometimes appearing of the size of one of the fixed stars, sometimes equalling the diameter of Venus, or even of the sun or moon; and they exhibit phases like those of the moon; their tails are supposed to arise from the gross atmospheres by which they are surrounded, driven off by the extreme heat of the sun, as these tails are in a direction opposite to that luminary, extending or shortening, as they approach toward or recede from it, their tails being a little incurvated, and most so near the ends of the tails, towards the parts, which the comets heads in their progress have left; the increased curvature towards the end of the tails is accounted for, from the diminished velocity with which the vapours ascend from the sun, in places more remote from the heated body of the comet; these tails are so thin, that stars can be seen through them.

The periods of the comets which have been observed, are supposed to be from seventy-five to five hundred and seventy-five years; there are but few of them whose orbits seem to be ascertained with accuracy; the period of one, which appeared in 1680, is supposed to be 575 years. The period of one which appeared in 1758, is thought to be about 75 years.

The next heavenly objects which arrest our attention, are the fixed stars, which are distinguishable from the planets, by being more luminous, and by continually exhibiting that appearance, which is called their scintillation or twinkling; which is usually ascribed to their appearing so extremely minute, that the interposition of the numerous small bodies, which are continually floating in the atmosphere, deprives us of the sight of them; but as they are continually changing their places, the stars become quickly again visible, which occasions the scintillation. Another remarkable property of the fixed stars, and that which first gave them their name, is their never changing their apparent situation with respect to each other. They are supposed by astronomers to be bodies of the same nature as our sun, having systems of planetary bodies revolving about them; their apparent diameters are so small, that very powerful telescopes but little augment their apparent magnitudes, and it is owing to their extreme brilliancy, that they are so clearly visible; they have been distinguished by astronomers into different orders according to their apparent magnitude, the largest being said to be, *of the first magnitude;* the next, *of the second magnitude,* and so on. Those of *the sixth magnitude,* are such as can be barely distinguished by the naked eye. Those which can only be seen by the aid of telescopes, are called *telescopick stars.* They have also been assorted by astronomers into different imaginary figures, called *constellations.* A part of the heavens, called the *galaxy* or *milky way,* is thought to owe its brilliancy to the vast number of very small fixed stars, with which it is crowded.

A small variation of about 20″ in the situation of the fixed stars, has been lately discovered, owing to the difference of the time, which their light takes to arrive at the earth, in different parts of her orbit, which is called the *Aberration of light.* It had long before been discovered, by the eclipses of Jupiter's moons, that light takes about 8 minutes in coming from the sun to the earth.

Having premised thus much concerning those heavenly bodies, which are the subject of astronomy; I proceed to deliver some propositions respecting them; in the course of which, the phenomena mentioned in the beginning of the elements of natural philosophy, as laws of the planetary motions, will be cited, as there laid down, Pl. L. denoting, *planetary law.*

PROPOSITION I. THEOREM.

That the forces, by which the primary planets are perpetually drawn from rectilineal motions, and are retained in their orbits, tend to the sun; and are reciprocally as the squares of their distances from its centre.

The former part of the proposition is manifest from the 1st planetary law, and prop. 2. Nat. Ph: and the latter part, from the 3d planetary law, and Cor. 6. prop. 3. Nat. Ph, as also from the 2d planetary law, and prop. 5. Nat. Ph.

Scholium. The same reasoning applies, to prove the same thing, to the satellites or moons, which revolve about Jupiter, Saturn, Herschel and Hercules, all the three planetary laws being applicable to their motions round their primaries. Note, that, when revolving bodies are spoken of, the laws are applicable to their centres of gravity; and when a primary, with one or more satellites, revolves about the sun, the laws are to be understood, as applicable to the centre of gravity of the whole revolving system.

PROP. II. THEOR.

That the force, by which the moon is retained in her orbit, tends to the earth ; and is reciprocally as the square of its distance from the earth's centre.

The former part of the proposition is manifest from the 1st planetary law, as applied to the moon. The latter part is deducible from the second planetary law, as applied to the moon, and prop. 5, Nat. Ph ; and also from comparing the centripetal force, by which the moon is retained in her orbit, which may be done by Cor. 9. prop. 3. Nat. Ph, with the force of gravity at the earth's surface. Assuming the moon's mean distance from the earth, as 60 of the earth's semidiameters, the lunar period with respect to the fixed stars, to be completed in 27d. 7h. 43m. as is determined by astronomers, and the circumference of the earth to be 132,192,000 English or American feet, as it has been estimated by geographers; if the moon were supposed to be deprived of all motion, and to be let down so, that, all that force urging it by which it is retained in its orbit, it should descend towards the earth, it would, in the space of one minute, by falling describe about $16\frac{1}{12}$ ft. Whence, since that force, in approaching to the earth, is increased in an inverse duplicate ratio of the distance, and therefore at the surface of the earth is 60×60 times greater than at the moon, a body, falling by that force in our regions, would describe in one minute $60\times60\times16\frac{1}{12}$ feet, and in the space of one second $16\frac{1}{12}$ feet, as is known to be the case; and therefore, by the 1st and 2d rules philosophizing, mentioned in the beginning of the elements of Nat. Ph. the force by which the moon is retained in its orbit, is the same, as that which we are accustomed to call gravity; for if gravity were different from it, bodies in falling towards the earth by both forces jointly, would descend with double velocity, and would describe in one second $32\frac{1}{6}$ feet, entirely contrary to experiment.

Cor. Hence, seeing that the revolutions of the primary planets round the sun, and of the secondary round their respective primaries, are phenomena of the same kind, as the revolution of the moon round the earth, and therefore, by rule 2, depend on causes of the same kind, especially since the forces, on which those revolutions depend, tend to the centres of the sun and primaries, and vary by the same law, as that by which the force of gravity does in approaching to and receding from the earth; and, since reaction is equal to action, the sun and primaries gravitate towards the planets, which revolve about them; and in short all planets gravitate towards each other. And hence Jupiter and Saturn, near their conjunction, disturb each others motions, the sun disturbs the lunar motions, and the sun and moon disturb our sea, thereby causing the tides.

PROP. III. THEOR.

That the axes of the planets are less than the diameters, which are perpendicular to them.

For by the circular motion of the planets on their axes, it happens, that the parts about the equator, by their centrifugal force, endeavour to recede from the axis, and thereby increase the equatorial diameter. Thus the axis of Jupiter is found to be less than his equatorial diameter. For the same reason, unless our earth was higher under the equator than at the poles, the seas at the poles would subside, and by ascending near the equator, would inundate the parts there.

Scholium. From the attractions of the sun and moon, on the elevated parts about the equator, arises the retrograde motion of the axis of the earth about the pole of the ecliptick, which has been mentioned.

And, though the motion of the planetary bodies in ellipses, the centre of motion being in a focus, is put among the planetary laws, being discovered by Kepler by most accurate observations on the planet Mars, as may be found in Small's excellent tract on Kepler's discoveries, a work well worthy the attention of the curious in astronomy, and is made use of in proving the law of the planetary attraction; yet as that law is deducible from Cor. 6. prop. 3. Nat. Ph. it from thence follows by Cor. 1. prop. 7. Nat. Ph. that the orbits must be ellipses, unless so far as these orbits may be a little disturbed by the mutual attractions of the planets on each other. Thus is this law corroborated by many concurrent proofs. It may be observed, that the planet Mars, from the greatness of its eccentricity, which may appear from the above planetary table, to be much greater than that of any of the other planets there mentioned, except Mercury, and from its proximity to the earth, appears to be peculiarly well adapted for observations of this kind.

Thus have I finished what I intended to deliver respecting the motions of the heavenly bodies; it being my intention to give only the general and most important laws, on which their motions depend, a brief account of those motions, with the demonstrations of the principles necessary for this purpose. Those who wish to go more fully into this subject, are referred to Newton's Mathematical principles of natural philosophy; to the understanding which work, it is hoped the information given in this book will be a great assistance.

NOTES.

Definition 1. *Book* 1. *of Euclid's Elements.*

What a point, line and superficies are, may be most easily conceived from the nature of a solid or body; for the bounds of a solid are not parts of it, and therefore have no thickness, their only dimensions therefore are length and breadth, they are therefore superficies or surfaces; but the bounds of those have only length, for if they had breadth, they would be parts, not bounds, they are therefore lines; whose bounds want even length, and have therefore no dimensions, and are points.

Ax. 10, 11 *&* 12. *B.* 1. *Eu.*—These three axioms, depending on definitions, are manifestly different from the other axioms. They have been differently managed by different editors. For the reasons of the mode in which they are here managed, see notes on 4. 1. and 29. 1. Eu.

Prop. 1. 1. *Eu.*—The proof given in this prop. of the circles intersecting each other, seemed quite necessary, as the intersection of the circles is requisite to the construction, and in geometry nothing should be assumed, except the axioms and postulates.

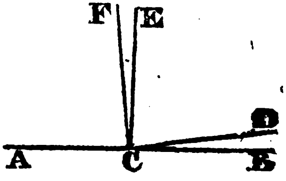
Prop. 4. 1. *Eu.*—The demonstration of this proposition has produced much disquisition; some have thought a postulate necessary, for removing one of the triangles about which the proof is exercised, and placing it on the other; but this does not appear to be requisite. There are, as far as I know, but two principles, whereon to found correct demonstrations of the equality of magnitudes, namely, by definition and coincidence, an instance by definition is found in the circle, all radiuses of the same circle being equal by def. 10. 1. which principle is used in each of Euclid's 3 first propositions, but the principle could evidently do but little.

There remains then the principle of coincidence, which Euclid uses in this proposition, and a most perfect one it appears to be, carrying with it the clearest evidence. That he might confine himself to principles laid down, the 8th axiom is used, that things, which being applied to each other, do coincide or agree, are equal; the application is not mechanical, it is altogether the work of the mind; the definitions of the terms, about which this proposition is exercised, are most clear and perfect, the axioms made use of most manifestly following from them, and the evidence of the coincidence of the figures so defined, on mental application, most clear, complete and satisfactory; and it appears to have been the best Euclid could possibly do.

The 11th axiom of this book is used in the proof of this proposition, and seems quite necessary, for if, part of the equal right lines supposed to be applied to each other coinciding, it were possible, that part of them should diverge or deviate from each other, or that two right lines should have a common segment, the demonstration of this proposition would be defective; and the impossibility of this being proposed to be demonstrated by Mr. R. Simson and other editors, subsequently to this 4th proposition, is a concession, that it ought not to be assumed in that proposition. The equality of all right angles to each other is proved, by means of this 11th axiom, in the theorem at the 11th of this book.

In most of the editions of Euclid's Elements which I have seen, the equality of right angles to each other is substituted for the 11th axiom here used, but I think there is reason to suspect, that the elements have in this instance been vitiated by Theon or some unskilful editor; the axiom here employed is used by Clavius and several others.

Many editors have attempted to deduce this principle from that of the equality of right angles, the demonstrations of two of them, Mr. R. Simson and Mr. Elrington are as follow:



Mr. R. Simson, *Cor. Prop.* 11. B. 1. If possible, let two right lines ACB and ACD have the segment AC common to both of them. From the point C draw CE at right angles to AB; and because ACB is a right line, the angle BCE is equal to the angle ECA (*Def.* 20. 1.); in the same manner, because ACD is a right line, the angle DCE is equal to the angle ECA; wherefore the angle DCE is equal to the angle BCE, the less to the greater, which is impossible; therefore two right lines cannot have a common segment.

Mr. Elrington, Note to Ax. 11. B. 1, which axiom asserts the equality of right angles to each other. Let, if possible, two right lines CD and CB, see the above fig., have a common segment AC, and let CE be perpendicular to the right line ACD, and if it be also perpendicular to the right line ACB, the angles DCE and BCE are equal (*Ax.* 11.), which is absurd. But if not, let CF be perpendicular to the right line ACB, and the angles ACF and ACE are equal (*by the same*), which is also absurd.

Of the first of these demonstrations, Mr. Elrington observes, that it does not appear to him perfect; for through the point C, it draws CE perpendicular to AC, and assumes that there can be but one perpendicular at that point; but this cannot be conceded, because, that a perpendicular may be raised, AC must be first produced, and if this could be done in different ways, there would be different perpendiculars at the point C, as appears from the construction of *Prop.* 11. B. 1. and therefore the whole demonstration fails.

And the second demonstration does not appear to me to be perfect, introduced at the axioms, or any where before *Prop.* 11. B. 1; because it supposes, that a perpendicular may be drawn to a given right line, from a given point therein; which is not taught before that proposition.

Since therefore *Ax.* 11. B. 1. here used, appears to be necessary to the proof of this 4th *Prop.* is very manifest from the definition of right lines, does not appear to be legitimately deducible from that usually used instead of it, namely, that all right angles are equal to each other, previously to this 4th *Prop.* which latter principle also, contrary to Euclid's usual practice, supposes the existence of right angles, before the possibility of their existence is shewn by any construction, and since moreover the axiom here used has been adopted by several respectable editors, and there is great reason to suppose that Euclid's Elements have been in many instances vitiated, is seemed quite expedient to follow the course here taken.

Prop. 22. B. 1. A like observation, as is made in *Prop.* 1. of this book, respecting the propriety of proving that the circles intersect each other, applies here also.

Prop. 29. B. 1. The axiom used in most editions of Euclid's Elements, instead of that which is the 12th in this, is, that "if "a right line, falling on two right lines, make the two interior

" angles on the same side of it, together less than two right " angles; these right lines may be so produced, towards the part, " on which the interior angles are less than two right angles, as " to meet:" which is acknowledged by Mr. R. Simson, Mr. Elrington, and many other editors, to be a proposition requiring demonstration, and indeed, before the reader could know, that, when the interior angles on one side were less than two right angles, the right lines would at all approach on that side, he should know, that the four interior angles were together equal to four right angles, and of course that the two interior angles formed at either intersection of the cutting line were equal to two right angles, which is not demonstrated until *Prop.* 13. B. 1. There is therefore reason to suspect, that the elements have in this axiom been vitiated, and the alteration here made appeared to me very expedient.

Prop. 44. B. 1. In most of the editions of Euclid's Elements which I have seen, in the construction of this problem, it is required, to make a parallelogram equal to the given triangle, having an angle equal to the given angle, and one of its sides in a right line with the given right line; for which as there is no previous problem or postulate, I cannot avoid being of Mr. Elrington's opinion, that the elements have probably been here vitiated, and have therefore altered the construction so as to avoid that irregularity. A like observation is applicable to some propositions in the sixth book.

Prop. 8. B. 2. This proposition, being of little use, and when requisite, easily supplied by other propositions, and never used in any subsequent part of this work, may be omitted at the discretion of the reader or teacher.

A like observation is applicable to the two subsequent propositions, which are rather curious than useful.

Prop. 11. B. 2. Professor Leslie calls the division of a right line in the manner taught in this proposition, the *medial section*, and therefore a right line so divided, may be said to be *cut medially*.

Prop. 13. B. 3. The demonstration here given of this proposition differs from that, which is in most editions of Euclid's Elements, being similar to that given by Mr. Elrington, for reasons assigned by Mr. R. Simson and Mr. Elrington in their notes on this proposition.

Def. 3. B. 5. Since, according to this definition, ratio is a certain relation between two magnitudes of the same kind, with respect to quantity; those writers of Navigation and Surveying, who, in their canons, compare lines with angular denominations, appear to be incorrect. This irregularity is avoided by Mr.

M'Kay and Mr. Gomere in their treatises on Navigation and Surveying. A like observation is applicable to the rule of three in Arithmetick, the irregularity being corrected by Mr. Stephen Pike and some others, in their treatises on that subject.

Def. 5 *and* 7. B. 5. If any two magnitudes whatever of the same kind were commensurable to each other, that is, had a common measure, and were to each other, as a number to a number; the definition of proportional magnitudes might be much simplified, and made similar to that of proportional numbers, in the 7th book of Euclid's Elements: but since, in several instances, magnitudes have been found by geometricians to be incommensurable to each other, as has been demonstrated of the diagonal and side of a square in the 117 Prop. B. 10. of Euclid's Elements, and of other magnitudes elsewhere; it was necessary to define proportional magnitudes, by properties applicable both to those which are commensurable and incommensurable; such is that by equimultiples, used by Euclid in his 5th book; and such is that by equisubmultiples, used in this work; of which mention is made more fully in the preface of this book.

And though ratio is a relation between two magnitudes of the same kind, yet the two first terms of four proportionals may be of a different kind from the two last.

Theor. 1. *at Prop.* 3. B. 5. Though this theorem and the following have been inserted for the purpose of demonstrating the following prop. being the 4th of this book, that none of Euclid's propositions might be omitted; yet as that 4th prop. is unnecessary in this place, on the plan used in this work, and is easily deducible from subsequent propositions of this book; both that proposition and these theorems may be omitted at discretion.

Prop. 15. B. 5. Euclid in this proposition cites *Prop.* 7. 5, instead of that which is demonstrated in *Cor.* 1. 7. 5. of this book.

Prop. 20 *and* 21. B. 5. Though these propositions are inserted, because they are in Euclid's Elements, yet being put there for the purpose of demonstrating the two following propositions, and not being necessary for that purpose on the plan used in this work, they may be omitted at discretion.

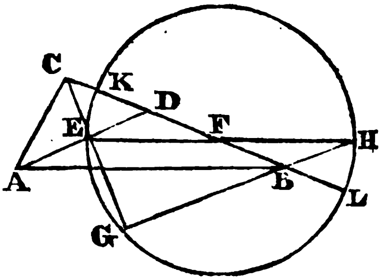
Prop. 26, 27, 28, *and* 29. B. 6. These propositions, being of little use, may be omitted at discretion; if omitted, the last construction only of the following 30th proposition should be used.

Theor. 1 *and* 2. *Prop.* 33. B. 6. These theorems are inserted for the purpose of bringing to one general principle, sundry demonstrations ad absurdum; they are used for this purpose in the 78th and 79th propositions B. 1. Sup. and by their means the demonstrations of several propositions of the 12th B. of Euclid's Elements may be much abbreviated.

Case 4. *Prob.* 2. *Plain Trigonometry.* This case may be solved, without letting fall a perpendicular, by means of the following

PROPOSITION.

The rectangle under the legs of a plain triangle, is to the rectangle under the half sum of all the sides, and the excess of that half sum above the base, in a duplicate ratio of radius to the cosine of half the vertical angle.



Let ABC be a plain triangle, of which AB is the base; the rectangle under AC and BC, is to the rectangle under the half sum of AB, BC and AC, and the excess of that half sum above AB, in a duplicate ratio of radius to the cosine of half the angle ACB.

Let CB be the greater of the legs AC and CB, and take thereon CD equal to AC; join AD, which bisect in E, and join CE; draw EH parallel and equal to AB, join HB, which produce to meet CE produced in G.

Because the triangles CEA and CED are mutually equilateral, the angle ACE is equal to DCE, and the angles AEC and DEC are also equal, and therefore right.

And since EH is equal and parallel to AB, BH is equal and parallel to AE (33. 1. *Eu*).

And in the triangles DFE and BFH, the angles at F are equal (15. 1 *Eu*), the angle FED is equal to FHB (29. 1. *Eu*), and ED to BH, being each equal to AE; therefore EF is equal to FH, and DF to FB (26. 1. *Eu.*); also the angle EGH is equal to CED (29. 1. *Eu.*), and therefore a right one; therefore a circle described from the centre F, at the distance FE or FH would pass through G (*Cor.* 31. 3. *Eu.*); let this circle be described, and meet CB produced in K and L.

Because both DC is to CE, and BC to CG, as radius is to the cosine of the angle DCE or of half the angle ACB (1 *Pl. Tr.*), the rectangle BCD is to the rectangle GCE, or its equal (*Cor.* 1. 36. 3. *Eu.*), the rectangle KCL, as the square of radius is to the square of the cosine of half the angle ACB (23. 6 and

22. 5. *Eu.*); but the rectangle BCD is equal to the rectangle ACB under the legs, because CD is equal to AC (*constr.*); and CL is equal to half the sum of the sides AB, BC and AC, because FL is equal to FH, the half of EH or AB; CD is the half of AC and CD together, and DF the half of DB, and therefore CF the half of AC and CB together; also CK is equal to the excess of that half sum above KL, the diameter of the circle, and therefore equal to EH or AB. Therefore the rectangle ACB, under the legs, is to the rectangle under the half sum of AB, BC and AC, and the excess of that half sum above AB, as the square of radius is to the square of the cosine of half the angle ACB, or, which is equal (20. 6. *Eu.*), in a duplicate ratio of radius to the cosine of that angle.

The application of this proposition, to finding any angle, as ACB, from all the sides given, is similar to that of the 30th *Sph. Tr.* to find an angle of a spherical triangle, from all the sides given, being the second solution of the 5th case of oblique angled spherical trigonometry, the application being thus. The rectangle under AC and CB : rectangle under $\frac{AB+BC+AC}{2}$ and $\frac{AB+BC+AC}{2}-AB$: : the square of radius : the square of the cosine of half the angle ACB (*by this Prop. and* 20. 6. *Eu.*)

Def. 1. *B.* 1. *Supplement.* Since the name *Conick Sections*, given by the ancients to the figures treated of in this book, is derived from their formation by the section of a solid, and the moderns have very generally fallen into the mode, of defining them from their description in a plain, it seems proper, that they should have a name different from that derived from the solid; of this Sir Isaac Newton seemed to be aware, when, after having demonstrated, that according to the laws of motion, bodies in free spaces must describe one of these figures, in the twentieth and nine following propositions of the first book of his principia, he calls them *trajectories*; but as the term trajectory is applicable to any line whatever described by a body moving according to any law, and with any resistance whatever, it appeared proper, to give them another appellation; and as they have been generally, by those who define them from their description in a plain, described by pins; the name of *Pattalloid*, derived from the Greek word "passalos" or "pattalos", which signifies a pin, has been selected.

Prop. 14. *B.* 1. *Sup.* From this proposition may be deduced, the law of variation of the square of the segment of a tangent, or rectangle under the segments of a secant, to a conick section or opposite sections, passing through a given point, and meeting a directrix, between that point and the section or sections, with the variation of the inclination of the tangent or secant to the directrix.

For the better understanding which, and some other things in this work, it seems proper to observe, that, as it is well known, that the area of a rectangle is found by the multiplication of the sides into each other, and therefore, if the area of a rectangle be divided by one of its sides, the quotient gives the other; mathematicians in their reasonings, often use the words *drawn into*, and *applied to*, instead of *multiplied* and *divided by*.

And the ratio of any two quantities to each other, is very conveniently expressed, by the quotient of the consequent divided by the antecedent; thus, the ratio of 1 to 3 may be expressed by $\frac{3}{1}$ or 3, being triple, and a ratio of greater inequality; the ratio of 3 to 1, by $\frac{1}{3}$, being subtriple, and a ratio of less inequality.

Which being premised, by this 14th *Prop.* see figures to it, $\frac{\text{rect. PKQ}}{\text{rect. SKT}}$ is $= \frac{KD^2}{KG^2} \times \frac{\text{diff. sq. KG and KM}}{\text{diff. sq. KD and KM}}$; but KX being radius, KD and KG are cosecants of the angles KDX and KGX (2. *Pl. Tr.*), and KM is to KX in the determining ratio; therefore the rectangle PKQ is, as the square of the cosecant of the angle KDX, applied to the difference of the squares of that cosecant and of a right line which is to radius in the determining ratio; or, the sines of angles being inversely as their cosecants (*Cor. 7. Def. Pl. Tr.*), inversely as the sine of the angle (KDX), which the secant or cutting line makes with the directrix, drawn into the difference of the squares of the same sine, and of a right line, to which radius is in the determining ratio. A like reasoning is applicable to the tangent KR.

Prob. 2. Solutions of the cases of Sph. Tr. In case 1 part 2, the affection of the angle ABC is ambiguous, unless it can be determined by this rule, that according as AC+BC is greater or less than 180°, A+ABC is greater or less than 180° (12. *Sph. Tr.*).

The following propositions are useful in removing ambiguities in the first solution of the 5th case.

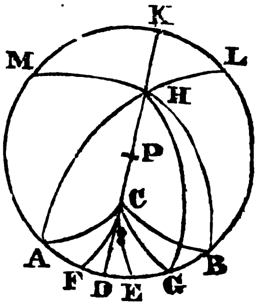
PROP. I. THEOR.

In an isosceles spherical triangle, the angles at the base are of the same affection as the sides.

If DE or DF, see fig. to prop. 15. Sph. Tr. be one of the sides of the isosceles triangle, the other side must meet the arch AKB in its continuation (15 *Sph. Tr.*); and if both be greater than DK, the sides are greater than quadrants, and the angles at the base obtuse (*by the same and Cor.* 2. 15. *Sph. Tr.*); if less, the sides are less than quadrants, and the angles at the base acute (*by the same*).

PROP. II. THEOR.

If the two least sides of a spherical triangle be of the same species, the perpendicular let fall from the angle included by them on the third side, falls within the triangle; or if the third side be less than either of the others, (these others being of the same species,) but greater than their supplements, the perpendicular falls within the triangle: and the less segment of the base and less vertical angle, are adjacent to the greater side, and the greater segment and greater vertical angle to the less side, if the sides be together greater than a semicircle.



Part. 1. Let ABC be the triangle, AC and CB being both less than quadrants, CD a perpendicular on its base AB; make AE equal to AC and BF to BC; the triangles AEC and BFC are isosceles, therefore the angle AEC is of the same species as AC, and BFC as BC (*by the prec. prop.*); and AC and BC are of the same affection (*Hyp.*); therefore the angles AEC and BFC are of the same affection; therefore the perpendicular CD falls between them (20 *Sph. Tr.*), and of course within the triangle ABC.

Part 2. In the triangle AHB, where AH and HB are each of them greater than AB; the two least sides HL and HM of the triangle MHL being the supplements of AH and BH, are each of them less than AB (*Hyp.*); but the arches ABL and BLM are semicircles (1 *Sph. Tr.*), taking from each the common part BL, the arches AB and LM are equal; therefore HM and HL are each of them less than ML, and being of the same species, the perpendicular HK falls between them, and therefore the perpendicular HD falls between AH and HB (*by part* 1). Let AD be less than DB, the angle AHD is less than BHD, and AH greater than BH, if AH and BH together be greater than 180°, or the point H be more remote from AB than its pole P (15. *Sph. Tr.*); for if the point H were not more remote from AB than that pole, neither AH nor BH would be greater than 90° (*Cor.* 2. 15. *Sph. Tr.*), nor therefore their sum greater than 180°.

Cor. 1. A perpendicular being let fall on a side of a spherical triangle considered as its base, from the opposite angle, the sides are both greater or both less than the perpendicular.

For since CD is perpendicular to AB, it passes through its pole P (*Cor.* 6. 2. *Sph. Tr.*); and if the vertex C be below the pole P, and therefore CD less than a quadrant, it is the least of all arches, which can be drawn from C to AB, and therefore less than CA or CB (15 *Sph. Tr.*); if the vertex of the triangle be above P as in H, the arch HD is the greatest of all arches, which can be drawn from H to AB, and therefore greater than HA or HB; if the perpendicular fall without the triangle, as in BGC or BGH; the demonstration is similar. In all cases therefore, the sides are both greater or both less than the perpendicular.

The case is omitted, when the pole P of the base being the vertex of the triangle, both the sides and perpendicular are quadrants (2 *Sph. Tr.*).

Cor. 2. When the perpendicular falls without the triangle, there are two perpendiculars, the less next the less side, and less than a quadrant or either of the sides; and the greater next the greater side, and greater than a quadrant or either of the sides; and either of them may be considered, as the proper perpendicular on the base produced.

In the triangle BGC, either CD or CK may be esteemed the perpendicular on the base GB produced, the less CD, next the less side CG, and less than a quadrant or either of the sides CG or CB; and the greater CK, next the greater side CB, and greater than a quadrant or either of the sides CB or CG; and either of them may be used for determining the several parts of the triangle BGC, but the one is sometimes more convenient than the other.

PROP. III. THEOR.

If to the base of a spherical triangle, a perpendicular be drawn from the opposite angle, which either falls within the triangle, or is the nearest of the two which fall without; the least of the segments of the base is adjacent to the least of the sides of the triangle, or to the greatest, according to the sum of the sides is less or greater than a semicircle.

See fig. to the prec. prop.

Part 1. If both the sides AC and BC, BC being the greater, of the triangle ABC, be less than quadrants, the perpendicular CD on the base AB falling within; because AC is less than CB, the arch AD is less than DB (15 *Sph. Tr.*), and so the perpendicular CD is adjacent to the less side AC.

Part 2. If both the sides CG and CB of the triangle CGB, be less than quadrants, the perpendicular CD, on the base BG produced falling without; the side CG adjacent to the perpendicular CD is the less (15 *Sph. Tr.*), and so the perpendicular CD is adjacent to the less side CG.

Part 3. If both the sides AH and BH, AH being the greater, of the triangle AHB, be greater than quadrants, the perpendicular HD on the base AB falling within; the segment AD adjacent to the greater is less than DB (15 *Sph. Tr.*), and so the perpendicular HD is adjacent to the greater side AH.

Part 4. If both the sides HG and HB of the triangle GHB be greater than quadrants, the perpendicular HD on the base BG produced falling without; the side HG adjacent to the perpendicular HD is the greatest (15 *Sph. Tr.*), and to the perpendicular HD is adjacent to the greater side HG.

Part 5. If the sides CG and HG be of different affections, CG being less and HG greater than a quadrant, both together being less than a semicircle, and GD be a perpendicular on the base HC produced; because CG and HG are together less than a semicircle (*Hyp.*), CG is less than the supplement of HG, therefore CD is less than the segment of CD produced, between D and the point in which CD produced would meet HG produced (15 *Sph. Tr.*), and so the perpendicular GD is adjacent to the less side CG.

Part 6. If the sides CB and BH be of different affections, CB being less and BH greater than a quadrant, both together being greater than a semicircle, and DBK be a perpendicular on the base CH produced, meeting it in D and K; let BH be produced to meet BK in M; and because CB and BH are together greater than a semicircle (*Hyp.*), CB is greater than the supplement HM of BH; whence BD and MK being each of them supplements of BK, and therefore equal, and both CB and HM less than quadrants, HK is less than CD (15 *Sph. Tr*), and therefore the perpendicular DBK is adjacent to the greater side BH.

Cor. Hence, all the sides of a triangle being given, it is easy to know, to which of the sides, including the angle from which a perpendicular is drawn to the opposite side, the perpendicular is adjacent; which is useful in the 1st solution of the 5th case of oblique angled spherical trigonometry, the segments of the base computed there, being those cut off by the nearest perpendicular.

Lemma 1. *Nat. Ph.* This lemma and the ten following, contain Newton's method of first and last ratios, and were, he said, inserted, that he might avoid the tediousness of deducing perplexed demonstrations ad absurdum, in the manner of the ancient geometers.

Prop. 3. *Nat. Ph.* In order to render the corollaries to this proposition more easily intelligible to the reader, I have thought proper to throw some of them into an algebraick form.

Let then F, denote the centripetal force of a body supposed t describe a circle; V, its velocity; P, its periodick time; a R, the radius of the circle: and let the corresponding small le ters denote the like in a body describing any other circle. Le the ratios be denoted in the manner mentioned in the note to 14. 1. Sup. namely, by the quotient of the consequent divided by the antecedent, in a fractional form. Then,

Number 1. Since the velocities, are as the arches described together; substituting the velocities for these arches, by the proposition $\frac{F}{f}$ is $=\frac{V^2}{v^2}\times\frac{r}{R}=\frac{rV^2}{Rv^2}$, which is the 1st. *Cor.*

2. The periodick times are in a ratio compounded of the direct ratio of the radiuses, and the inverse one of the velocities, that is, $\frac{P}{p}$ is $=\frac{R}{r}\times\frac{v}{V}=\frac{Rv}{rV}$.

3. Since, by No. 1, $\frac{F}{f}$ is $=\frac{rV^2}{Rv^2}$, or multiplying each term of the last quantity by Rr, $=\frac{r^2V^2R}{R^2v^2r}$, or substituting for $\frac{r^2V^2}{R^2v^2}$, its equal (by inverting and squaring the terms of the equation of No. 2), $\frac{p^2}{P^2}$, $=\frac{p^2R}{P^2r}$, which is the 2nd. *Cor.*

4. If the periodick times be equal, or $\frac{P}{p}$ be $=1$, the velocities are as the radiuses, as is manifest; but it follows also from No. 2, for by that $\frac{Rv}{rV}$ is $=\frac{P}{p}=1$ (*Hyp.*); therefore Rv is $=$rV, and, by 16. 6. *Eu*, V : v : : R : r.

5. The same thing being supposed, the centripetal forces are as the radiuses. For, by No. 1, $\frac{F}{f}$ is $=\frac{rV^2}{Rv^2}$, and dividing each term of the last quantity by rV and Rv, which are equal by No. 4, $\frac{F}{f}$ is $=\frac{V}{v}$, or, which is equal by the same No, $=\frac{R}{r}$. The 3d *Cor.* is included in this No. and the preceding.

In like manner, as the 3d. *Cor.* is proved in the two preceding numbers, may the truth of the 4th, 5th, 6th, and 7th corollaries be shewn, of which it seems sufficient to give an example, in the proof of the 6th, which is done in the two following numbers.

6. If the periodick times be in a sesquiplicate ratio of the radiuses, the velocities are in an inverse subduplicate ratio of the radiuses. For, by inverting the terms of No. 2, $\frac{rV}{Rv}$ is $=\frac{p}{P}=\frac{r^{\frac{3}{2}}}{R^{\frac{3}{2}}}$(*Hyp.*), therefore, dividing each quantity by $\frac{r}{R}$, $\frac{V}{v}$ is $=\frac{r^{\frac{1}{2}}}{R^{\frac{1}{2}}}$.

7. The same thing being supposed, the centripetal forces are inversely as the squares of the radiuses. For, by No. 3, $\frac{F}{f}$ is $=\frac{p^2R}{P^2r}$, or, $\frac{p^2}{P^2}$ being $=\frac{r^3}{R^3}$(*Hyp*), $=\frac{r^3R}{R^3r}$, and, dividing each term of the last quantity by Rr, $=\frac{r^2}{R^2}$. The 6th *Cor.* is included in this No. and the preceding.

As to *Cor.* 9. Let a, denote an indefinitely small arch described by a body moving in a circle; d, the diameter of the circle; and s, the sagitta which the body in falling, would describe in the time of the description of the arch; and, by the proof of the proposition, as a^2 is $=ds$; let t represent any time whatever, and multiplying each term of the equation by t^2, a^2t^2 is $=ds\times t^2$, and therefore, by 17. 6. *Eu*, $d : a\times t :: a\times t : s\times t^2$, but $a\times t$ represents the arch described, and $s\times t^2$, the descent of the body with the same centripetal force, in the time t; whence appears the truth of the corollary.

THE END.

SUBSCRIBERS' NAMES.

A

Abercrombie, Rev. James
Adams, Hon. John Quincy
Adlum, John, Esq.
Alexander, William, Esq.
Allen and Hill, Messrs.
Allen, Ebenezer N. Esq.
Allen, Rev. Ethan,
Allen, Matthew J. M. D.
Allen, Richard, Esq.
Allen, Richard N. Esq.
Allen, Robert, M. D.
Allen, Robert T. M. D.
Allen, William, Esq. 4 copies
Allen, William H. Esq.
Allender Joseph, M. D.
Archer, John, M. D.
Archer, Stevenson, Esq.
Ashman, J. Esq.

B

Baker, Samuel, M. D.
Barber, John F. Esq.
Barber, L. P. M. D.
Barleve, L. P. M. D.
Barney, John, Esq.
Barroll, Wm. H. Esq.
Barry, Rev. Edm. D. D. 6 co.
Bartow, Rev. John V.
Battee, Rd. H. Esq.
Bayne, Rev. Thomas
Belt, George G. Esq.
Bend, Wm. B. Esq.
Billingslea, James, Esq.
Blackburn, George, A. M.
Bland, Hon. Theodorick
Boggs, Alexander, Esq.
Boisseau, D. W. Esq.
Bond, Edw. F. Esq.
Bond, Joshua, Esq.
Bond, Thomas, W. Esq.
Bowie, Thomas, Esq.
Bowie, Thomas H. Esq.
Bowie Wm. Esq.
Boyd, Andrew, Esq.
Boyle, James, Esq.
Bradford, Samuel, Esq.
Brent, Lieut. Ab.
Brewer, John, Esq.
Brewer, Nichs, jr. Esq.
Brice, Jas. T. Esq.
Brice, Hon. N.
Brown, George, M. D.
Brown, Matthew, Esq.
Brown, Samuel, jr. Esq.
Brownley, Joseph, Esq.
Bruti, S. Esq.
Bryan, Wrightson, Esq.
Bryson, James, Esq.

C

Cannel, Isaac, Esq.
Cantwell, Simon, Esq.
Carroll, J. H. Esq.
Carrol, Michael J. Esq.
Chalmers, James, Esq.
Chandler, Jehu, Esq.
Chapman, Henry H. Esq.
Chase, Samuel, Esq.
Chatard, Peter, M. D.
Chauncey, Geo. Esq.
Chevigne, L. Prof. Mat. S. M. C.

Claggett, Thomas, Esq
Claggett, Rt. Rev. T. J. Bp. Md
Clarke, James, Esq
Claude, Dennis, M. D.
Claxton, Christopher, Esq
Clay. Revd J. C.
Clenaghan, Wm M. Esq
Cleveland, A. B. Esq
Coale, Edw. J. Esq
Cohen, J. I jun. Esq
Coleman, Revd John
Cooper, Revd J. G.
Corner, James, Esq
Cotter, Richard, Esq
Coulter, John, M. D.
Cox, Hugh, Esq
Crabb, R. J. Esq
Craig, John D. Esq 10 co.
Cross, Walter, Esq
Curran, Barn. Esq
Curran, Philip, Esq

D

Dallam, Francis J. Esq
Dalrymple, Jas. A. D. Esq
Dashiell, Revd George
Davidge, Francis H. Esq
Davidge, John B. M. D.
Davis, Elijah, M. D.
Davis, Revd Henry L.
Davis, James G. Esq
Davis, Reuben H. Esq
De Butts, Elisha, M. D.
Deem, Jacob, Esq
De Le Ree, Revd James
Dennis, Littleton, Esq
Donaldson, Samuel J. Esq
Dorsey, Henry, of Edw. Esq
Dorsey, Hon. Walter
Drill, Christian, Esq
Ducatel, Edme, Esq
Duke, Revd William
Duncan, Revd William
Dunn, Robert, Esq
Duvall, Lewis, Esq
Duvall, William, Esq

E

Edes, Col. Benjamin
Egerton, Charles C. Esq
Evatt, John, Esq
Everitt, William B. Esq

F

Farrell, Thomas, Esq
Foard, Joseph R. Esq
Foley, Thomas, Esq
Forman, Revd John
Fowler, Baruch, Esq
Foxcroft, William, Esq
Franklin, Thomas, Esq
Frazier, Henry, Esq
Freeman, William H. Esq
French, Ebenezer, Esq
Frick, William, Esq
Frisby, Richard, Esq

G

Gadsden, Revd C. E.
Gardiner, Thomas, Esq
Garner, Henry, Esq
Garretson, James, Esq
Gassaway, John, Esq
Gibson, John Lee, Esq
Gibson, William, M. D.
Gibson, William, Esq
Gill, R. W. Esq
Gilmore, Robert, Esq
Glendy, Revd John
Glenn, John, Esq
Gould, James F. Esq
Gowan, John, Esq
Graham, J. Esq
Graves, Revd Richard
Green, Jonas, Esq
Green, Wm S. Esq
Green, Zachariah, Esq
Gregory, Henry, Esq

Grey, Horatio N. Esq
Grey, Revd James, D. D.
Griffith, Edw. Esq
Grosvenor, John, Esq
Guyton, Benj. Esq

H

Hagarty, John, Esq
Hale, Henry, Esq
Hall, Edw. Esq
Hall, J. E Esq
Hall, Robt. Esq
Hall, Richd W. M. D. 3 co.
Hall, Sergeant, Esq
Hall, Thomas, P. M. D.
Hall, Wm W. Esq
Hanson, Hon. Chas. W. 3 co.
Hargrove, Revd John
Harlan, Jeremiah, Esq
Harris, John, Esq
Harwood, Benj. Esq
Harwood, Frederick, Esq
Harwood, H. H. Esq 2 co.
Harwood, Richard, Esq
Hawkins, John, Esq
Hawkins, Matthew, Esq
Hawley, Revd Wm
Hays, Henry H. B. Esq
Heard, Benj. J. Esq
Heath, James P. Esq
Heath, Upton S. Esq
Helfenstein, Revd Albert
Henderson, John, Esq
Henry, William, Esq
Henshaw, Revd John P. K.
Herbert, John C. Esq
Hewett, Daniel, Esq
Hicks, Revd Galen
Hillary, Tilghman, Esq
Hinckley, Edward, Esq
Hoffman, David, Esq
Hollingsworth, Thomas, Esq
Hollingsworth, Samuel, Esq
Hopkins, Ephraim, Esq
Hopkins, John, Esq
Hopkins, Joseph, Esq
Hopkins, Samuel, Esq
Howard, Henry, Esq
Howard, Henry, of Jno. Esq
Howard, John B. Esq
Howard, William, M. D.
Hughes, Christopher, Esq
Hughes, Jeremiah, Esq
Hughes, John, Esq
Hull, Revd James F.
Hulse, Isaac, Esq
Hunter, James, Esq
Hurd, Ebenezer, Esq

J

Jackson, Edward, Esq
Jackson, William, Esq
Jameson, Rush, Esq
Jay, Thomas, Esq
Jennings, Samuel K. M. D.
Iglehart, James, jun. Esq
Inglis, Revd James, 2 co
Johnson, Edward, Esq
Johnson, John, Esq
Johnson, Revd Matthew
Johnson, Thomas, M. D.
Johnson, James, Esq
Jolley, Edward, Esq
Jolley, William D. Esq
Jones, M. C. Esq
Jones, Talbot, Esq
Jones, Thomas, Esq
Judah, Revd H. R.

K

Keith, Revd Reuel
Kell, Thomas, Esq [2 co.
Kemp, Rt. Revd Jas. Bp. of Md
Kent, Joseph, M. D.
Kesley, Revd William
Key, Francis, Esq
Keyser, Samuel, Esq
King, Charles B. Esq

King, Levin R. Esq
Knight, Nathaniel, Esq
Kurtz, Revd Daniel D. D.

L

Lance, Revd Maurice H.
Latimer, Stephen, Esq
Law, Anthony, Esq
Learned, Joseph D Esq
Lindon, Revd John
Little, Francis, Esq
Lockerman, Richard, Esq
Lockerman, T. R. Esq
Love, John, Esq
Lux, William, Esq

M

M'Causland, George, Esq
M'Comas, James S. Esq
M'Cormick, Revd And. T.
M'Crodden, James, Esq
M'Dowell, John, D. D.
M'Dowell, Maxwell, M. D.
M'Elfresh, John H. Esq
M'Elhiney, Revd George
Macgill, Robert, Esq
M'Intire, David, Esq
M'Kenny, John, Esq
M'Kim, John, Esq
M'Math, William, Esq
Mackubbin, George, Esq
M'Parlin, William, Esq
M'Pherson, Henry H. Esq
Madden, Cato P. B. M. Esq
Maddox, Samuel, Esq
Magee, John, Esq
Magruder, Alexander C. Esq
Magruder, John R. Esq
Magruder, Richard A. C. Esq
Magruder, William, Esq
Maguire, Hugh, Esq
Marlow, Thomas, Esq
Marriott, Wm. H. Esq
Mason, James M. Esq
Matchett, George, Esq
Maulsby, Israel D. Esq
Maynadier, Henry, Esq
Meade, Revd William
Meconekin, William E. Esq
Meredith, Jonathan, Esq
Mitchell, G. C. Col. U. S. A.
Monks, John, Esq
Montgomery, John, Esq
Moore, George, Esq
Moore, Jason, Esq
Moore, John L. Esq
Morrison, David, Esq
Mosher, James, Jun. Esq
Munroe & French, Messrs
Munroe, his Ex. James, P. U. S.
Munroe, John, Esq
Munroe, Thomas, Esq
Murray, Daniel, Esq
Murray, Henry M. Esq

N

Nevins, Rev. William
Ninde, James, Esq
Ninde, Revd William
Norris, Nelson, Esq
Norris, Revd Oliver
Norris, Samuel, Esq

O

O'Brien, Revd Timothy
Ogden, Jos. J Esq
Osborne, Len. U. S. Navy
Owen, John, M. D.

P

Patterson, William P. Esq
Pattison, Granville Sharp, M. D.
Periman, Isaac, Esq
Pfeiffer, Revd Henry H.
Phillips, Isaac, Esq
Phillips, William, Esq
Picket, John W. Esq
Picknell, Joseph, Esq

Pigman, B. T. Esq
Pilmore, Revd James
Pindle, Benjamin, Esq
Pinkney, Jonathan, Esq
Pinkney, Ninian, Esq [2 co.
Pinkney, Hon. Wm. Sen. U. S.
Pleasants, John P. Esq
Posey, Lawrence, Esq
Potter, Nathaniel, M. D.
Power, Michael, Esq
Price, E. Esq

R

Rafferty, Revd William
Ramsay, James, Esq
Raymond, D. Esq
Read, Revd Thomas
Reardon, Doctor James
Reed, Jas. Capt U. S. Art.
Reese, Revd Danl E.
Reese, D M. M. D.
Reese, J. J. Esq [C.
Reynolds, Owen, Prof. Mat. B.
Reynolds, Samuel, Esq [10 co.
Richards & Mallory, Messrs
Richardson, Robert, Esq
Ricketts, Samuel Esq
Ridout, Horatio, Esq
Ridgely, Charles, Esq
Ridgely, John, M. D.
Roberts, Revd George, D. D.
Robinson, Henry, Esq
Rogers, Micajah, Esq
Rogers, Solomon A. Esq
Rogers, Thomas, Esq
Rogerson, Thomas, Esq
Ruff, Rd. H. Esq
Rummells, Mr. Stephen

S

Sands, Joseph, Esq
Schultz, Jacob. Esq
Schwarer, George, Esq
Scott, Mr. Leonard
Selman, John, Esq
Semmes, A. Esq
Sewell, Charles S. Esq
Shaw, James, Esq
Sherburne, Jonathan, Esq
Shryer, Daniel, Esq
Sinclair, Revd Wm. D. D.
Smith, Clement, M. D.
Smith, Paca, Esq
Smith, Revd Purnell F.
Sorrel, Francis, Esq
Spurrier, Beale, Esq
Steele, John F. Esq
Steever, Daniel, Esq
Stephen, Hon. John
Stevens, Revd Daniel
Stevenson, Cosmo G. M. D.
Steward, Isaac, Esq
Steward, Nathan, Esq
Stewart, Caleb, Esq
Stewart, Richard, Esq
Stewart, Wm. Esq
Stockett, R. G. M. D. [town 2 co.
Stopford, Jas. Earl of Cour-
Stopford, Rt. Rvd. Bp of Cork,
Stran, Thomas P. Esq [Ireland.
Stull, O. H. W. Esq
Stump, Henry, Esq.
Sullivan, Daniel, Esq.

T

Taylor, Wm. W. Esq
Thomas, D. L. Esq
Thompson, George H. Esq
Thompson, John, Esq
Tuck, Washington G. Esq
Turner, Alexander, Esq
Turner, Philip, Esq
Turner, Samuel B. Esq
Turner, Revd Samuel H.
Tyng, Revd Stephen H.
Tyson, John S. Esq
Tyson, Philip T. Esq

V

Vansicle, Bennett, Esq
Vinson, Charles, Esq

W

Walker, Joseph, Sen. Esq
Walker, Joseph, Jun. Esq
Walker, Revd Joseph R.
Walker, Mrs. Mary
Wall. Spencer, Esq
Wallace. James, Esq
Walsh, John, Esq
Walsh, R. of Benj. Esq
Warfield, William, Esq
Waters, Amos, Esq
Watkins, N. J. Esq
Watkins, Tobias, Esq
Webster, Henry W. M. D.
Webster, Toppan, Esq
Weems, William L. Esq.
Weller. Revd George
Wells, William, Esq
West, Arthur P. Esq
West, Stephen B. Esq
Wever, Caspar W Esq
White, John C. M. D.
Williams, Barrett, Esq
Williams, George, Esq
Williams, Revd Geo. M.
Williams. Nathaniel, Esq
Willis, Henry, Esq
Willis, Jesse H. Esq
Wilmer, Revd Wm. H.
Winchester, Geo. Esq
Winchester, Samuel, Esq
Woodward, Henry, Esq
Worthington Samuel, Esq
Wyatt, Revd Wm. E.

Y

Yeiser, John, Esq
Yhary, Alexander, Esq
Young, Revd Noble
Young, Robert, Esq.